The Teaching of Arithmetic

FIFTIETH YEARBOOK, PART II

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THE TEACHING OF ARITHMETIC

Prepared by the Society's Committee

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EDITOR'S PREFACE

The suggestion that the National Society publish a second yearbook on arithmetic was presented to the Board of Directors in a communication prepared by Professor Buswell after consultation with several other professional students of the subject. It was noted that the theoretical concepts underlying the discussion of various problems of learning and instruction as presented in the Twenty-ninth Yearbook have been significantly modified by the results of more recent investigations in the psychology of learning. Professor Buswell's statement stressed the further point that changing conceptions of the social values of arithmetic and of the relation of arithmetic to other subjects of instruction have new implications for classroom procedures and for the professional preparation of teachers and supervisors of arithmetic. The Board of Directors indicated their interest in receiving additional information regarding the kind of yearbook that would be most useful at this time.

Professor Buswell then formulated a plan for a yearbook based on current ideas of the nature and uses of arithmetic and with due consideration for the need of further exploration of the values of new theories of learning and new materials of instruction for the improvement of teaching practices in this subject. The outline for such a yearbook was reviewed by the Board of Directors at its meeting in May, 1948. It was finally approved at the ensuing meeting, and Professor Buswell was requested to serve as chairman of the yearbook committee.

The objectives of the committee are clearly set forth in the introductory chapter. In the opinion of the editor, this committee has been notably successful in guiding the productive efforts of the contributors toward the effective expression of these purposes in their discussion of both the theoretical and the practical aspects of the topics assigned them. It may reasonably be assumed that the yearbook will serve the useful purpose of contributing to a better understanding of the next steps in progress toward the solution of the still perplexing problems related to the teaching of arithmetic.

NELSON B. HENRY

TABLE OF CONTENTS

	PAGE
OFFICERS OF THE SOCIETY FOR 1950-51	iii
THE SOCIETY'S COMMITTEE ON THE TEACHING OF ARITHMETIC	v
ASSOCIATED CONTRIBUTORS	v
EDITOR'S PREFACE	vii
CHAPTER	
I. INTRODUCTION	1
G. T. BUSWELL	
Aims of the Yearbook	2
The Organization of the Yearbook	4
II. ARITHMETIC IN THE ELEMENTARY-SCHOOL CURRICULUM	6
ERNEST HORN	
The Scope of Arithmetic in the Elementary School	6
The Place of Arithmetic in Various Curriculum Patterns	8
The Number and Difficulty of Arithmetical Abilities and Concepts Required in Other Fields	9
The Incidence of Arithmetical Terms in the Texts and References in Other Areas	10
The Ability of Pupils To Manage Mathematical Concepts	12
Contribution of Arithmetic to Other Fields	14
Contributions of Other Fields to Arithmetic	15
No Defense of Formal Instruction	21
III. THE NATURE AND SEQUENCES OF LEARNING ACTIVITIES IN ARITHMETIC	22
HARRY G. WHEAT	
Introduction: The Arithmetic We Teach	22
The Nature of the Learning Activities	25
The Sequences of the Learning Activities	39
IV. ARITHMETIC FOR PRESCHOOL AND PRIMARY-GRADE CHILDREN	53
ESTHER J. SWENSON	
Introduction	53
Child Growth and Development in Arithmetic	54
Content and Teaching Method in the Early Years	58
Concluding Statement	75

CHAPTER PAGE

V. ARITHMETIC IN THE MIDDLE GRADES 76
C. L. THIELE
Critical Nature of Arithmetic in the Middle Grades 76
The Problem of Method 78
A Point of View toward the Arithmetic Content of the Middle Grades. 81
Summary 101

VI. ARITHMETIC IN THE JUNIOR-SENIOR HIGH SCHOOL 103
H. VAN ENGEN
Introduction 103
Factors Influencing Junior-Senior High School Arithmetic 104
Basic Concepts Pertaining to Junior-Senior High School Arithmetic 106
The Senior High School Program 115

VII. LEARNING AND TEACHING ARITHMETIC 120
HERBERT F. SPITZER
Common Initial Approaches to Instruction 120
Analysis of Instructional Procedures 128
Other Features of Learning and of Teaching Methods 135
Use of Tests in Instruction 137
Special Learning Aids 138
Problem-solving 139
Theoretical and Experimental Considerations 140
Concluding Statement 142

VIII. THE PSYCHOLOGY OF LEARNING IN RELATION TO THE TEACHING OF ARITHMETIC 143
G. T. BUSWELL
Changing Concepts in the Psychology of Arithmetic 143
Classification of Theories of Learning 144
Influence of Association Theories 145
Influence of Field Theories 146
Effects of Maturation 153
Conclusion 154

IX. INSTRUCTIONAL MATERIALS FOR TEACHING ARITHMETIC 155
FOSTER F. GROSSNICKLE, CHARLOTTE JUNGE, AND WILLIAM METZNER
I. Relation of Instructional Materials to Learning 155
II. Illustrations of the Use of Instructional Materials 164
III. Visual and Manipulative Materials 172

CHAPTER PAGE

X. TESTING INSTRUMENTS AND PRACTICES IN RELATION TO PRESENT CONCEPTS OF TEACHING ARITHMETIC 186
HERBERT F. SPITZER
Outstanding Characteristics of Present Concepts of Teaching 186
Current Testing Instruments and Practices 187
Arithmetic Tests in Recent Publications 194
Recommendations 196
A List of Arithmetic Tests 200

XI. THE TRAINING OF TEACHERS OF ARITHMETIC 203
FOSTER E. GROSSNICKLE
Requirements for Certification 203
Questionnaire Study of Training of Teachers of Arithmetic in Teachers' Colleges 205
Recommendations 229

XII. MATHEMATICAL BACKGROUND NEEDED BY TEACHERS OF ARITHMETIC 232
C. V. NEWSOM
Introduction 232
The Historical Development of Arithmetic 235
The Real Number System 236
The Arithmetic of Measurement 244
Further Attention to Applications 246

XIII. IN-SERVICE DEVELOPMENT OF TEACHERS OF ARITHMETIC 251
D. BANKS WILBURN AND G. MAX WINGO
The Concept of Supervision 251
Studying the Content of Arithmetic 253
Applying Theories of Learning to the Teaching of Arithmetic 255
Assisting Teachers To Initiate Learning Activities Designed To Improve Instruction in Arithmetic 256
Meeting Local Needs in Developing Programs for Improving Instruction in Arithmetic 261
Summary 268

XIV. THE SOCIAL POINT OF VIEW IN ARITHMETIC 269
B. R. BUCKINGHAM
Arithmetic Evolves from the Emerging Needs of Human Society 269
The Effects of Arithmetic on Social Institutions 272

CHAPTER PAGE

The Effect of Social Institutions on Arithmetic 273
An Arithmetic Theory for the Schools of Today 278

XV. NEEDED RESEARCH ON ARITHMETIC . 282
G. T. BUSWELL
Research and the Improvement of Instruction in Arithmetic 282
Proposals for Research on Problems of Teaching and of Learning in Arithmetic 285

INDEX 299
INFORMATION CONCERNING THE SOCIETY 303
LIST OF PUBLICATIONS OF THE SOCIETY 305

CHAPTER I

INTRODUCTION

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Arithmetic exhibits some marked contrasts when compared with some of the other content areas of education. Unlike chemistry, physics, and the social sciences, its content is not subject to radical changes due to discoveries such as those involving new elements, forms of energy, or ideas of social organization. The basic content of arithmetic is old and is unlikely to change rapidly in any major fashion. Arithmetic still operates through the media of a zero and nine other digits. It deals in the main with ideas of increase and decrease through processes of addition and multiplication, on the one hand, and subtraction and division on the other. It operates with whole numbers and with fractions, including the special kind of fractions that we call per cents. It deals with ideas of ratio and proportion. It uses these processes for various kinds of measurement and with different kinds of denominate numbers. This content and these processes are no different today than they were when the first school was organized in America and, of course, far earlier than that. Why, then, are there still unanswered questions that warrant a new yearbook on arithmetic?

The content of arithmetic is old because it is inherently governed by the number system under which it operates. The number system is coherent and internally consistent. Two plus two cannot equal five and still be a part of the decimal system. The *system* determines the relations and processes that are possible. However, as a part of the total school enterprise, new emphasis is being given to certain important ideas in the field of arithmetic.

a) There are new ideas as to the ways to use arithmetic and as to the relative importance of these ways. Most schools in 1900 were quite content if the outcome of teaching arithmetic was an increase in computational ability. This competence was described as a contribution to social efficiency. Without ascribing less value to computational proficiency, schools of the present are not satisfied unless the outcomes include also a high degree of competence in quantitative thinking, much of which

carries on with computational approximations and with no pretense of pencil and paper exactness.

b) There are new ideas as to ways to learn arithmetic. The application of memorized rules that are not understood, which characterized the teaching of arithmetic at the time of the American Revolution, has been replaced by an emphasis on understanding which has made the phrase "meaningful arithmetic" a common one in today's schools.

c) There are new ideas as to the way the study of arithmetic should be related to other parts of the school's program. These ideas range from a willing acceptance of a subject organization, in which arithmetic is one of the "subjects," to ideas of curriculum organization in which arithmetic is scarcely recognized by name but in which number relations and quantitative ideas permeate all of it. The structure of the curriculum, when and *if* it again crystallizes into a period of stability, may be strikingly unlike the structure present concepts would lead one to predict, but, whatever the organization of content may be, there is no escape from the necessity of learning number processes and relationships.

d) There are new ideas about methods and materials of instruction. Improvements made possible through new manipulative aids and through new possibilities of demonstration by the use of sound films are altogether desirable.

e) Also, there are new ideas regarding arithmetic which are due to changes in concepts of educational values. A different use of arithmetic characterizes life in an age of scientific methods from that which prevails when educational values are more completely centered in personal-philosophical problems. Furthermore, the arithmetic which fits best an objective of social efficiency may not satisfy equally an objective of developing personal adjustments and feelings of security. Arithmetic contributes to both social and personal needs, and these change from generation to generation. They are different at different levels of maturity for the same individual. Arithmetic must contribute to the growing child in terms of his expanding personal and social experiences.

Another yearbook on arithmetic is justified, not so much because the problems with which it is concerned are new as because many outcomes which for decades have been desired are still not reached. It is only the easier problems in the field of arithmetic that have been solved by research; the more difficult still resist our efforts.

Aims of the Yearbook

The committee in charge of the present yearbook hopes to make three main contributions to the teaching of arithmetic.

First among these is an emphasis on the importance of arithmetic in

any type of elementary-school curriculum. A major part of the thinking in our society is quantitative thinking, and its quality is conditioned by society's competence in understanding and dealing with number relations. The committee believes that competence in quantitative thinking is of first-order importance in education and that it warrants purposeful teaching of the most effective type that can be devised. The committee recognizes that such purposeful treatment may be accomplished in various ways, but under no circumstances can unplanned, incidental teaching be considered one of the effective ways of teaching arithmetic. Whatever the type of curriculum in the elementary school, one of the important criteria for evaluating it is the effectiveness with which number relations are taught. Arithmetic's potential contribution to the other parts of the curriculum and to the needs of society outside the school warrant direct and purposeful teaching.

A second major point of emphasis by the committee is that, while the number system and number processes are the heart of arithmetic, they are important only as they function in the actual thinking of people. There is no excuse for arithmetic in a social vacuum. Arithmetic is a product of thought and is concerned with the mathematics of social thinking. There is no dichotomy requiring a choice between mathematical and social arithmetic. These are two aspects of the same thing. The present yearbook deals with both aspects. It gives more than usual emphasis to the mathematical meanings inherent in arithmetic, but only because a better understanding of these meanings contributes to clearer quantitative thinking and to greater effectiveness in social situations. It is time that the argument over mathematical arithmetic versus social arithmetic be consigned to the resting place of other useless controversies in education which have wasted energy that could have been employed in more productive ways.

The third major point of emphasis in the yearbook is on the better preparation of teachers of arithmetic, both through preservice and in-service training. As will be shown in later chapters, the present lack of teacher preparation in this field is alarming. In no other major curriculum field are the opportunities for scholarly preparation so meager. The arithmetical understandings of most teachers of arithmetic rest on no content beyond that covered in their own eighth-grade course in arithmetic. High-school and college mathematics contributes little of a specific nature to the teacher of arithmetic. The teachers' college courses deal with methods but seldom go beyond the eighth-grade level in mathematical content. The committee believes that better preparation of teachers of arithmetic is a prerequisite to increased effectiveness of the arithmetic program. It is the hope of the committee that this yearbook may mark

the initiation of an up-grading development in teacher preparation of nation-wide scope.

The Organization of the Yearbook

The chapters of the yearbook, following the Introduction, fall into a definitely planned organization.

In chapter ii, Horn deals with arithmetic from the point of view of the total elementary-school curriculum. This broad point of view is basic to the committee's belief that the importance of arithmetic is independent of the kind of curriculum organization through which a school operates.

Chapters iii through vi deal with the arithmetic program. In chapter iii, Wheat deals with the nature of learning activities in arithmetic and their sequence as determined by the relationships which are inherent in a decimal number system. In the three following chapters Swenson, Thiele, and Van Engen cover the arithmetic program in the primary grades, the middle grades, and the junior high school, respectively.

Chapters vii through x constitute a group dealing with problems of learning, instruction, and evaluation. In chapter vii, Spitzer deals with methods of teaching arithmetic that are most conducive to effective learning. This is followed by a chapter in which Buswell relates current theory in the psychology of learning to the teaching of arithmetic. In chapter ix Grossnickle, Junge, and Metzner provide an exceptionally comprehensive treatment of instructional materials and manipulative aids. In the fourth chapter in this group, Spitzer shows how testing instruments and practices must be modified to fit present concepts of the teaching of arithmetic.

Chapters xi through xiii deal with problems of training teachers of arithmetic. This problem is opened up in chapter xi by Grossnickle, who reports the results of a wide survey of current practices in training teachers and then discusses needed improvements in teacher-training programs. Chapter xii presents a concrete proposal covering the mathematical background needed by teachers of arithmetic. The committee invited Professor C. V. Newsom to prepare this chapter because it wanted the views of an outstanding mathematician. The proposals of Dr. Newsom may seem rigorous to some teachers, but the committee believes that the course which he outlines should constitute a goal for teacher preparation toward which consistent and determined progress must be made. The present gap between the arithmetical knowledge of teachers of arithmetic and that of pupils at the end of the eighth grade is shockingly small. The committee believes that competent teaching requires preparation that goes much beyond the common "review course" which so frequently represents the scholarship level of teachers of arithmetic. Our schools

would not tolerate teachers of English or of science whose academic preparation was limited to a "review course" of the content of high-school work in those fields; yet in the case of arithmetic we have been complacent about it. Beyond the level of preservice training, chapter xiii, by Wilburn and Wingo, presents a discussion plus concrete examples of desirable in-service training of teachers.

In chapter xiv, Buckingham gives a comprehensive treatment of the social point of view in arithmetic. This chapter furnishes an excellent setting for interpreting the total contribution of chapters ii through xiii.

The final chapter, xv, is a composite contribution of some twenty members of the profession to future research in arithmetic. In view of the marked change in the concept of teaching arithmetic, as compared with that represented in the Society's *Twenty-ninth Yearbook*, the committee believes that if research is to be effective it must deal with problems that relate to present concepts of teaching arithmetic. The substance of the chapter is a group of concrete proposals for such research. These proposals illustrate both a live awareness of the changes which have been taking place and, in some cases, a nostalgia for research on outworn issues which have lost the importance they once had. The committee hopes that the chapter may stimulate an interest in research on a meaningful concept of arithmetic which will result in improvements in the methods of teaching arithmetic and in the types of tests used for evaluating the results of teaching.

CHAPTER II

ARITHMETIC IN THE ELEMENTARY-SCHOOL CURRICULUM*

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This chapter does not deal primarily with the place of arithmetic as assessed by its life values but with its place and relationships in the total curriculum of the elementary school. The many unmistakable contributions of arithmetic to life have been convincingly portrayed in various yearbooks of the National Council of Teachers of Mathematics. It will be apparent, as the discussion in this chapter proceeds, that the writer suspects that the proponents of the organized course of instruction in arithmetic have been too modest and limited, rather than too extravagant and comprehensive, in their claims.

THE SCOPE OF ARITHMETIC IN THE ELEMENTARY SCHOOL

Throughout the following discussion, arithmetic is broadly defined to include all mathematical concepts and abilities appropriate to and needed by pupils in the elementary school. Its scope is helpfully outlined in Brownell's list of desirable arithmetic outcomes:

(1) Computational skill:
Facility and accuracy in operations with whole numbers, common fractions, decimals, and per cents. (This group of outcomes is here separated from the second and third groups which follow because it *can* be isolated for measurment. In this separation much is lost, for computation without understanding *when* as well as *how* to compute is a rather empty skill. Actually, computation is important only as it contributes to social ends.)

(2) Mathematical understandings
a. Meaningful conceptions of quantity, of the number system, of whole numbers, of common fractions, of decimals, of per cents, of measures, etc.
b. A meaningful vocabulary of the useful technical terms of arithmetic which designate quantitative ideas and the relationships between them.
c. Grasp of important arithmetical generalizations.
d. Understanding of the meanings and mathematical functions of the fundamental operations.

* The writer is indebted to Dr. Herbert F. Spitzer for many suggestions in preparing this chapter.

e. Understanding of the meanings of measures and of measurement as a process.

f. Understanding of important arithmetical relationships, such as those which function in reasonably sound estimations and approximations, in accurate checking, and in ingenious and resourceful solutions.

g. Some understanding of the rational principles which govern number relations and computational procedures.

(3) Sensitiveness to number in social situations and the habit of using number effectively in such situations;

a. Vocabulary of selected quantitative terms of common usage (such as kilowatt hour, miles per hour, decrease and increase, and terms important in insurance, investments, business practices, etc.).

b. Knowledge of selected business practices and other economic applications of number.

c. Ability to use and interpret graphs, simple statistics, and tabular presentations of quantitative data (as in study in school and in practical activities outside of school).

d. Awareness of the usefulness of quantity and number in dealing with many aspects of life. Here belongs some understanding of social institutions in which the quantitative aspect is prominent, as well as some understanding of the important contribution of number in their evolution.

e. Tendency to sense the quantitative as part of normal experience, including vicarious experience, as in reading, in observation, and in projected activity and imaginative thinking.

f. Ability to make (and the habit of making) sound judgments with respect to practical, quantitative problems.

g. Disposition to extend one's sensitiveness to the quantitative as this occurs socially and to improve and extend one's ability to deal effectively with the quantitative when so encountered or discovered.[1]

This broad definition seems to be in harmony with modern trends.[2]

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[1] William A. Brownell, "The Evaluation of Learning in Arithmetic," *Arithmetic in General Education*, pp. 231–32. Sixteenth Yearbook, National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1941.

An equally comprehensive list, although differing somewhat in some of the items included, is given by Brueckner in the Thirty-fourth Yearbook of the National Society for the Study of Education, *Educational Diagnosis*, pp. 269–71. Chicago: University of Chicago Press, 1935.

[2] See especially *The Teaching of Arithmetic* (Tenth Yearbook, 1935) and *Arithmetic in General Education* (Sixteenth Yearbook, 1941), National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University.

The ramifications of mathematical thinking are impressively portrayed in Roget's *Thesaurus* in the sections treating concepts of relation, quantity, variation, order, number, time, change, causation, dimensions, form, and motion.

Nor should it disturb those who are deeply concerned with the development of the skill in computation. As Buckingham points out:

> An arithmetic course which does not have as a very definite objective the development of ability to compute with accuracy and facility is wholly inadequate. However, the curriculum which tries to accomplish nothing but accuracy and facility in figuring is woefully one-sided. Moreover, a curriculum which sets out to do that and that only will not even serve its own purpose satisfactorily.[3]

The Place of Arithmetic in Various Curriculum Patterns

The place of arithmetic in the total curriculum is affected somewhat by differences in the curriculum pattern of different schools. The subject curriculums, which are by far the most frequent and widespread, vary from those which follow rather closely the textbooks in arithmetic and in other subjects to those which, although following the subject divisions in a broad way, place the emphasis on problems and activities. Other types of curriculum organization likewise exhibit considerable variety, including the experience unit, activities organizations, broad centers of interest, etc.

Actually it is very difficult to make a valid distinction between what are called experience units, activities, or centers of interest, on the one hand, and subject-matter units, on the other. The same problem or unit may appear under all types of organization. Many units listed as activities or experience units are nothing more than camouflaged sections of traditional subject-matter fields.

The relationships of arithmetic to other instructional areas are, however, much the same in principle, regardless of the general type of curriculum organization. They are determined by the nature of the learning undertaken in other fields. Most units or problems in the general areas of health, science, and social studies, if substantial and if vigorously attacked, involve arithmetical concepts and abilities, and the arithmetical aspects are often very crucial.

The consideration of the relation of arithmetic to other subject areas involves such questions as: What are the nature and amount of the demands which other areas make upon arithmetic? How adequate are the pupils' abilities to meet these demands? What are the probable or potential contributions of these areas to the development of arithmetical abilities? Is there a need for specially planned instruction in arithmetic? And, if so, what is the relation of that instruction to the arithmetical instruction incident to the attack on mathematical aspects of other areas?

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[3] *The Teaching of Arithmetic, op. cit.*, p. 52.

The Number and Difficulty of Arithmetical Abilities and Concepts Required in Other Fields

The demands made upon arithmetical abilities in the attack on problems in other instructional areas, as well as the difficulties which pupils have in meeting these demands, constitute serious problems. In discussing these problems, illustrations are here drawn chiefly from the social studies; first, for simplicity of treatment; second, because social studies constitute the core in most thoroughgoing, integrated plans; and third, because the writer has long been interested in the quantitative aspects of texts and references in the social studies and in the adequacy with which children deal with them. This emphasis upon arithmetic in the social studies should not, however, obscure its importance in other fields, including even music and art.[4]

Regardless of the curriculum pattern, it is entirely feasible to determine, for each field in the curriculum and at each grade level, first, what mathematical concepts and abilities are most frequently and crucially needed and, second, the extent to which pupils have developed them. There has been in the last twenty years a large number of investigations of one or both of these problems.[5]

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[4] See *The Place of Mathematics in Modern Education.* Eleventh Yearbook, National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1936.

[5] Clifford Woody, *Nature and Amount of Arithmetic in Types of Reading Material for the Elementary Schools.* Bureau of Educational Reference and Research, Bulletin No. 145. Ann Arbor: University of Michigan, 1932.

G. T. Buswell and Lenore John, *The Vocabulary of Arithmetic,* pp. 10–14, 107–16. Supplementary Educational Monographs, No. 38. Chicago: University of Chicago Press, 1931.

Edward E. Keso, "The Relationship of Mathematics and Geography," *School Science and Mathematics,* XLIV (October, 1944), 598–600.

Agnes G. Gunderson, "Nature and Amount of Arithmetic in Readers for Grades I and II," *Elementary School Journal,* XXXVI (March, 1936), 527–40.

Jean B. Tompkins and C. Newton Stokes, "Eight-Year-Olds Use Arithmetic," *Childhood Education,* XVI (March, 1940), 319–21.

Roy D. Willey, "A Study of the Use of Arithmetic in the Elementary Schools of Santa Clara County, California," *Journal of Educational Research,* XXXVI (January, 1943), 353–65.

Some of the most significant data have been reported in masters' theses but, unfortunately, have not been published. Examples are: Ruth H. Bishop, "Arithmetical Skills, Concepts, and Knowledges Involved in a Fourth-Grade Geography Textbook." Unpublished Master's Thesis, State University of Iowa, 1933; Teresa H. O'Donnell, "A Suggested List of Basic Reference Points Based on an Analysis of Fifth-Grade Textbooks." Unpublished Master's Thesis, State University of Iowa, 1941.

A critical evaluation of the data from these investigations, even though it is fragmentary, warrants the following conclusions: First, there is a very heavy demand upon arithmetical concepts and abilities in other school work. Second, there is a large number of mathematical concepts and abilities which are required in most other fields of study. Third, the ability of pupils to deal with quantitative statements in other fields is very limited.

The Incidence of Arithmetical Terms in the Texts and References in Other Areas

Every investigator has shown the incidence of arithmetical terms to be very large—how large depends upon how broadly "arithmetical terms" is defined. If indefinite and marginal terms are included, such as *more, heavy,* and *high,* the incidence shown in an analysis of recently published geography texts runs as high as one word in seven.[6] This is not surprising when one realizes that, of the first 1069 words in the list compiled by Thorndike and Lorge,[7] more than one in ten are reasonably specific arithmetical, geometrical, or statistical terms. And if indefinite mathematical terms are included, the proportion is about one in four. A large number of these mathematical terms appear frequently in the texts and references in the content fields.

Additional evidence on the high incidence and importance of mathmatical concepts in other fields is provided by Pressey.[8] The words in these vocabularies were secured, first, by having teachers list, from over a hundred textbooks, those words in the various subjects which, in their opinion, would give children difficulty, and, second, by having teachers of each subject check words deemed "absolutely essential to the subject"[9] or, if not essential, important. A very considerable proportion of the words rated as absolutely essential by teachers of mathematics and arithmetic are also deemed essential by teachers of other subjects. Among the words considered essential in art, for example, are *area, balance, breadth, circle, cube, depth, dimension, distance, horizontal, length, measure, parallel, perpendicular, rectangle, square, triangle,* and *unit.*

The difficulty of dealing with the mathematical concepts in reading is increased by the fact that they frequently appear in combination, as:

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[6] Samplings analyzed by the writer's seminar, January, 1950.

[7] Edward L. Thorndike and Irving Lorge, *The Teacher's Word Book of 30,000 Words.* New York: Bureau of Publications, Teachers College, Columbia University, 1944.

[8] Luella Cole Pressey, *The Special Vocabularies of the Public School Subjects.* Bloomington, Illinois: Public School Publishing Co., 1924.

[9] Luella Cole Pressey, "The Determination of the Technical Vocabulary of the School Subjects," *School and Society,* XX (July 19, 1924), 91–96.

"almost two hundred years"; "through many centuries"; "millions of *dollars"; "nearly two miles wide"; "ranges from twenty-five to one hundred twenty-five per square mile"; "nine hundred square miles"; "three* and a *half million"*; "(the river) *falls only four inches* in *a mile"*; ". . . . *less* than *twenty-four times* the *annual world production"; "about eighty thousand tons a month"; "each* dot stands for *100,000* people"; "*several thousand acres"; "five hundred* people *on each square mile"; "half a mile above sea level"; "hundreds* of *millions* of *board feet"; "average* rainfall *ranges from twenty to thirty inches"; "Each year from sixty* to *seventy per cent"*; ". . . . land *values doubled* and *doubled again"; "over 1,000,000 bales* of cotton *a year* which is *about one-fifth as much* as is *normally* exported by the United States"; "It is *estimated* that the *remaining* known deposits of oil in the United States *total some 21 billion barrels.*"

The ability to understand such statements as these involves much more than momentary, piecemeal attention, for each statement has relations to other facts in a larger setting, and the ability to deal with these relations is heavily conditioned by the pupils' grasp of the number system. In other words, it requires functional, quantitative thinking, which Hedrick has described thus:

> Wherever the question arises as to "*what*" affects "*this*," and as to just *how* it works, there is functional thinking. To contribute toward better thinking by more people in all such cases is the proper goal of all mathematical teaching in all schools.[10]

In many instances the quantitative or mathematical elements constitute the heart of the problem. In a large proportion of the units or problems in social and natural sciences and in health, as well as in special units, such as housing, safety, the airplane, or conservation, the ability to deal with the mathematical factors—not merely quantitative factors—is indispensable. These factors must not be neglected or slighted, regardless of the teacher's view on integration or on the advisability of a special curriculum in arithmetic. The thoughtful teacher of these units, impressed by the frequency and basic importance of the mathematical factors as well as by their interrelationships, is likely to welcome the efforts of the teacher of arithmetic to develop systematically the abilities required to deal with them.

Since quantitative statements appear so frequently, it may be argued that, if they are adequately dealt with as they occur, the essential arithmetical abilities will be developed. The *if* looms large. Quantitative statements *have* appeared frequently, yet in most schools the abilities needed

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[10] E. R. Hedrick, "Functional Thinking," *School Science and Mathematics*, XL (April, 1940), 361.

to deal with them *have not* been developed. This is not surprising in view of the formal and superficial methods of instruction which are widespread. The verbalistic quiz following the reading of a single textbook by all pupils is especially unproductive. For example, the pupils read: "The *average* population is *one hundred* persons *per square mile.*" Little or no contribution can be expected, either to arithmetic or to an understanding of density of population, when the teacher asks: "What is the average population per square mile?" and the pupils answer, "One hundred," merely parroting the statement in the book.

The Ability of Pupils To Manage Mathematical Concepts

It is to be expected that many mathematical concepts should, because of their complexity, cause pupils considerable difficulty. This has been found to be the case. Investigations have consistently shown serious shortcomings on the part of pupils to deal with such concepts successfully, either in isolation or in context.[11]

Buswell and John report the percentages of correct responses in Grades IV to VI for the following terms to be: acre, 38.9; area, 28.1; average, 43.7; difference, 71.3; rectangle, 33.7. More pupils thought an acre to be larger than a square mile than believed it to be smaller than a city block. Among the highly erroneous responses were: *acorns, 100,000 miles, man's name, houses, to have a stomach ache.*[12] The arithmetical terms tested were taken from elementary-school arithmetics. The authors recommend that increased attention be given to the development of understanding of common arithmetical terms.[13]

Ryan found that both definite and indefinite terms were difficult for children to interpret in the textbook situations and that the indefinite terms more frequently led to misunderstandings than did the definite terms. She reports that *ten square miles* meant to various pupils *about the size of Chicago, about the size of the state of Iowa, about the size of Washington Park, as large as ten acres, about one lot,* and *here to Key West in a straight line.*[14]

Indefinite quantitative terms, such as *a great deal, many, far, thick,* have been shown to be more difficult than are definite statements of quantity.[15] It is hard, even in the case of definite statements, for students

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[11] Ernest Horn, *Methods of Instruction in the Social Studies*, pp. 143, 189–93. New York: Charles Scribner's Sons, 1937.

[12] Buswell and John, *op. cit.*, chaps. iii and iv.

[13] *Ibid.*, chap. vi.

[14] Horn, *op. cit.*, pp. 190–91.

[15] Otto J. Gabel, "The Effect of Definite versus Indefinite Quantitative Terms upon the Comprehension and Retention of Social Studies Material," *Journal of Experimental Education*, IX (December, 1940), 177–86.

to approximate the author's meaning closely enough to make an intelligent judgment of quantity. Specific statements have the advantage, however, of giving the student something to work on. Indefinite terms afford little clue as to the amounts that the author had in mind. *One hundred* has at least one consistent meaning, whether it applies to dollars, people, or distances, although its significance in these three instances is, of course, different. We should not be surprised, therefore, when judgments of students are far afield. Bedwell reports that *a great deal* in the sentence, "We grow a great deal of flax in our country," meant to various students *ten bushels, ten thousand bushels,* and *seventy million bushels. Thick,* in the sentence "Most of Greenland is covered by a thick cap of ice and snow which never melts away," was estimated by various students to be *one inch, three feet, three thousand feet,* and *thousands and thousands of feet.*[16]

Students, especially in Grades VI to VIII, are frequently confronted by concepts which are much more subtle and troublesome than those cited above. They have special difficulty with statistical terms, such as *average, sample, normal, range, variation,* and *distribution,* which are fairly common in the textbooks used in those grades.[17] The importance of these concepts in dealing with quantitative data suggest that persistent efforts should be made for their development.[18]

Students show marked deficiencies in reading tables and graphs. Scott, in testing the ability of sixth- and eighth-grade pupils to read a table showing the relation of the amount of water supplied in irrigation to the yield of wheat, asked: "As more water was supplied to the land what happened to the yield of wheat per acre?" Among the answers were: "It would die." "Warsh away." "Grew more wheat than water supply." "Flooded it." "Larger families use more water than others." "It went up and so did the price." "There is more wheat than water."[19]

It was earlier pointed out that many quantitative statements, because of their complexity and their relation to the total context in which they appear, should not be considered in an isolated, piecemeal fashion. The same can be said of the difficulties that pupils have in dealing with these statements. All of the many concepts and abilities listed by Brownell in

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[16] Horn, *op. cit.*, pp. 190–91.

[17] Lucy Scott, "A Study of Children's Understanding of Certain Statistical Concepts in Social Studies." Unpublished Doctor's Dissertation, State University of Iowa, 1942.

[18] See the excellent discussion by Helen M. Walker in *Mathematics in Modern Life,* chap. viii. Sixth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1931.

[19] Scott, *op. cit.*

portraying the scope of arithmetic (pp. 6–7) are needed at least occasionally in Grades I to VIII, and several of them, indeed often many of them, may be needed in the interpretation of a single paragraph.

Contribution of Arithmetic to Other Fields

The frequency and difficulty of the demands made upon arithmetic in attacking problems in other subjects suggest that instruction in arithmetic can make important contributions to facilitating the work in these areas. The evidence from many investigations which have recently been completed, although dealing with limited areas, indicates that there is a large number of arithmetical abilities and concepts which are so commonly and persistently needed that the student should have them in stock. Examples rather consistently reported, in addition to arithmetical operations and the reading of numbers, are: *amount, area, average, center, circle, depth, difference, distance, equal, height, increase, length, line, number, part, per cent, proportion, quantity, scale, single, square, straight, time, total, vertical, weight, whole,* and *width.* These terms are only a small proportion of the fairly definite mathematical terms that appear with high frequency in the reading done in other fields. Indefinite quantitative terms are much more frequently found, especially at the lower grade levels. The more seriously the meaning theory of arithmetic is taken and the greater the emphasis upon the development of quantitative thinking, the better the argument for including, as a part of the instructional plan in arithmetic, the development of concepts and abilities needed to deal with both definite and indefinite terms.

It is especially important that pupils be helped to establish standard reference units and the habit of using these units in interpreting quantitative statements. These units are built partly on the experience in the immediate locality and partly on concepts already achieved. For example, a known area of approximately one square mile may be used to interpret other small areas, the area of one state may serve to interpret the area of a country, the height of a certain tree will assist in understanding a statement of the height of other trees. Obviously the unit to be utilized should be appropriate to the size or amount of the new quantitative statement to be interpreted. As Spitzer points out, the pupil who has an accurate idea of the distance "one foot" is not helped much in interpreting the distance "one mile" even though he may know that the mile contains 5280 feet.[20]

If the teacher of arithmetic will conscientiously develop the basic concepts and abilities listed by Brownell (pp. 6–7) and Brueckner to guide

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[20] Herbert F. Spitzer, *The Teaching of Arithmetic.* Boston: Houghton Mifflin Co., 1948.

instruction in that field, the pupil will be sensitized to the mathematical elements in problems in other fields and will be freed from the frustrations and annoyances which come from interrupting, sometimes for a considerable time, the attack on a problem until the necessary mathematical concepts and abilities can be developed. In the long run, this should save time and add to the vigor and efficiency of work in other fields. In emphasizing the contributions of arithmetic to other areas, however, it must be kept in mind that these are only a part of the responsibilities which the teaching of arithmetic must assume. As will later be pointed out, arithmetic has its own peculiar responsibilities, and these are determined primarily by the needs for mathematical computation, understanding, and thinking in life outside the school.

The conditions under which thinking is done in the attack on problems in other areas, as well as the types of thinking required, suggest that considerable emphasis be given to mental arithmetic. Mental arithmetic, however, should be broadly defined to include not merely practice on number combinations but also the making of approximations, the meaningful interpretation of quantitative statements, the acquiring of mathematical vocabulary, including both definite and indefinite mathematical terms, and the habit of relating quantitative statements to reference points in the students' experience. This suggestion is, of course, not new. In a stimulating article on mental arithmetic in *A Cyclopedia of Education* (1914), David Eugene Smith concluded: "In general, however, it may be said that mental arithmetic offers the best means for correlating the subject with the pupils' other work both within and without the school."

Of course, as he elsewhere points out, instruction should not be limited to mental exercises. Pencil and paper computations and mental arithmetic are complementary and reinforcing in their relationships.

Contributions of Other Fields to Arithmetic

No one can seriously question the fact that a considerable amount of arithmetic is learned in connection with the study of problems in other areas. Indeed, if the arithmetical demands in other fields are met, as recommended above, marked contribution from these fields is inevitable not only to the motivation for learning arithmetical abilities but also to their development and maintenance. The achievement of pupils in arithmetic in investigations reported by Harding and Bryant, Hizer and Harap, Harap and Mapes, Seay and Meece, Williams, Wrightstone, and others compares favorably with achievements under traditional teaching.[21]

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[21] Lowry W. Harding and Inez P. Bryan, "An Experimental Comparison of Drill and Direct Experience in Arithmetic Learning in a Fourth Grade," *Journal of Edu-*

It is impossible to tell from the published reports, with one or two exceptions, just how mathematical concepts and abilities were dealt with when needed, how much time was spent, what methods were used, or what provisions, if any, were made for growth and maintenance. Many schools allegedly depending on incidental learning actually do a considerable amount of systematic teaching of arithmetic which either duplicates or closely resembles the practices recommended under the meaning theory.[22] In addition, some "bootlegging" is not unknown on the part of teachers who are skeptical of the degree to which incidental learning can be depended upon. The amount of arithmetic that is learned incidental to the study of other school subjects depends upon the nature of the content in those areas, the adequacy of the treatment of the mathematical aspects, and the way in which mathematical concepts and abilities are developed in connection with the problems to which they apply. If the curriculum in other areas is designed to meet the most important needs of life, including both the present and future needs of pupils, and if the essential mathematical aspects of these needs are adequately dealt with, the potential contributions to arithmetical ability are, of course, enhanced. Hanna and his associates, after teaching certain units, concluded, "The functional experience of childhood is not alone adequate to develop arithmetical skills."[23] However, the kind and number of the computational problems reported seem to fall short of what a vigorous attack on the units would require. The experience units reported by Williams were saturated with arithmetic, if not at times overloaded, and, in addition, arithmetic periods were regularly scheduled in which the learning, usually related to the experience units, was highly motivated and meaningful.

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cational Research, XXXVII (January, 1944), 321–37; Irene S. Hizer and Henry Harap, "The Learning of Fundamentals in an Arithmetic Activity Course," *Educational Method,* XI (June, 1932), 536–39; Henry Harap and Charlotte E. Mapes, "The Learning of Fundamentals in an Arithmetic-Activity Program," *Elementary School Journal,* XXXIV (March, 1934), 515–25; Maurice F. Seay and Leonard E. Meece, *The Sloan Experiment in Kentucky* (Bureau of School Service, Bulletin No. 4, Vol. XVI. Lexington: University of Kentucky, June, 1944); Catharine M. Williams, "Arithmetic Learning in an Experience Curriculum," *Educational Research Bulletin,* XXVIII (September, 1949), 154–62; J. W. Wrightstone, *Appraisal of Newer Practices in Selected Public Schools,* chap. ii (New York: Bureau of Publications, Teachers College, Columbia University, 1935).

[22] C. L. Thiele, "An Incidental or an Organized Program of Number Teaching?" *Mathematics Teacher,* XXXI (February, 1938), 63–67; G. T. Buswell, "Deferred Arithmetic," *Mathematics Teacher,* XXXI (May, 1938), 195–200.

[23] Paul R. Hanna and Others, "Opportunities for the Use of Arithmetic in an Activity Program," *The Teaching of Arithmetic, op. cit.,* pp. 85–120.

Some drill was also provided.[24] Harap and Mapes deliberately selected units which were rich in the application of certain fundamental processes. It is clear, however, that units in other fields should not be chosen primarily to develop mathematical abilities. On the contrary, they should be chosen chiefly, if not wholly, to meet the purposes of the areas in question. Conservation materials, for example, should be chosen because the understanding of the facts and their significance is necessary for intelligent citizenship. However, when the quantitative factors of a problem or unit are crucial, the grade placement of the unit may depend in part upon the degree to which pupils have perfected or can easily attain the abilities needed to deal with these factors.

There is no need to feel that if the needs in other fields are chosen primarily for their distinctive contributions in those fields, integration with arithmetic will be limited. The heavy incidence of mathematical terms in textbooks in other fields and in general reading suggests that, if the mathematical aspects of the problems at hand are to be adequately dealt with, it will be difficult for instruction in arithmetic to keep pace.

Some of the units which have been utilized as the center for integrating arithmetic seem somewhat superficial in conception or inappropriate for the grades where they were taught. If average achievement results from the study of such units, it seems not overly sanguine to expect much greater achievement with more important and better graded units. In general, the proponents of integration appear to overestimate and the opponents to underestimate the amount and quality of mathematical achievement under integrated plans. So substantial, however, are the potential contributions of other fields to arithmetic that, when achievement in arithmetic, as measured by an adequately descriptive test, is low, one should suspect that there is something wrong with the content and methods of instruction in other areas as well as with instruction in arithmetic.

There remains the question of how efficient it is, either from the standpoint of arithmetic or from the standpoint of achievement in other fields, to depend solely upon the development of mathematical concepts and abilities on the spot as they are needed. Is a price paid in other fields when the attack on problems is interrupted while the required arithmetical abilities are being acquired? Is arithmetic learned and retained more readily? Are gaps left among the arithmetical abilities needed in life? Is there any saving of time in the total program? Are abilities likely to be

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[24] For complete data see Catharine M. Williams, "The Contribution of an Experience Curriculum to Mathematical Learning in the Sixth Grade." Unpublished Doctor's Dissertation, Ohio State University, 1947.

developed to a high level of efficiency under integrated plans alone? Are interest in arithmetic and appreciation of its significance sufficiently fostered? And, most important of all, perhaps, do most teachers have the time and ability to develop competence in arithmetic under an integrated plan alone?

These questions are not rhetorical. They must be asked and answered by anyone who wishes fairly and critically to assess the place of arithmetic in the curriculum. It is not surprising that different persons, because of differences in experience with integrated plans as well as differences in the grasp of the problems of teaching arithmetic, should not always arrive at the same answers. And it must be admitted that, on the basis of experimental evidence, a confident answer to some of the questions cannot be given at this time. It is the unanimous opinion of the present committee, however, that, after a careful appraisal of such evidence as does exist, and after giving full credit to what has been or is likely to be accomplished under integrated plans, such plans by themselves cannot be depended upon to develop arithmetical concepts and abilities to the level and scope required in life.[25] An especially designed program of instruction in arithmetic is essential, and such a program should include not only provision for systematic and meaningful learning in the arithmetic class but also careful attention to the mathematical needs and contributions of other areas. There are several reasons for this conclusion.

First, an appraisal of the evidence on incidental learning from the psychological laboratory, from experiments in the classroom, and from ordinary experience suggests that the development of important abilities cannot be left safely to incidental learning. Arithmetic is no exception. Indeed, the fact that it is a system makes it peculiarly unlikely that the essential abilities and concepts, interrelated as they are, will be satisfactorily developed incidentally or by piecemeal accretion. The discovery and development of the number system is a major accomplishment of human endeavor, and it required centuries to bring it to its present efficiency. As Schorling points out, not even a bright child can be expected to achieve an efficient command of it merely through opportunistic learning associated with units and activities. "The basic ideas," he urges, "must be taught through numerous and vivid experiences, and we need a long time in which to do it if we are to place the emphasis on meanings

[25] See also William A. Brownell, "Psychological Considerations in the Learning and the Teaching of Arithmetic," *The Teaching of Arithmetic, op. cit.*, pp. 1–31; "The Second Report of the Commission on Postwar Plans: The Improvement of Mathematics in Grades I to XIV," *Mathematics Teacher*, XXXVIII (May, 1945), 195–221.

and on social implications, and if, finally, these basic concepts are to be tied together into a system of ideas or a way of thinking."[26]

The proponents of integrated curriculums sometimes insist that the learning of skills in such programs is not incidental in the narrow sense that the laboratory psychologists use the term to apply to learning which takes place without specific motive or specific instruction. This is, of course, true; for, when the pupils see the need for a given arithmetical computation in attacking a unit, they *are* motivated to learn and perform the computation, and, if they perform it successfully, they have specific practice. But, if the motivated practice stops with the satisfaction of the immediate needs, it stops too short. It should lead to and be supplemented by interest and work in arithmetic as such, and as a system. Williams, who placed unusually heavy emphasis upon the arithmetical aspects of the units which she taught and carried the more difficult problems into the scheduled arithmetic periods, reports that, as children became aware of their inabilities and understood their difficulties in mathematics, they requested more work in arithmetic.[27]

Second, arithmetic has its own values and its own responsibilities aside from the development of the ability to deal with the mathematical exigencies of other areas. Its content should be selected and organized chiefly in terms of its values in life. And these values cannot be determined by item-by-item, piecemeal appraisal which does not take into account the systematic relationships involved. The scope of the curriculum has been helpfully outlined by Brownell and by Brueckner[28] and their outlines have been supplemented in "The Second Report of the Commission on Postwar Plans." Special attention is called to two of the theses in that report: "Thesis 2: We must discard once for all the conception of arithmetic as a mere tool subject" and "Thesis 3: We must conceive of arithmetic as having both a mathematical aim and a social aim." Regarding Thesis 3, the Commission makes the following comments:

> The fundamental reason for teaching arithmetic is represented in the social aim. No one can argue convincingly for an arithmetic which is sterile and functionless. If arithmetic does not contribute to more effective living, it has no place in the elementary curriculum. To achieve the social aim of arithmetic, children must be led to see its worth and usefulness.
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> We may grant the paramount importance of the social aim, and yet insist

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[26] Raleigh Schorling, "The Place of Mathematics in General Education," *School Science and Mathematics*, XL (January, 1940), 19.

[27] Williams, "The Contribution of an Experience Curriculum to Mathematical Learning in the Sixth Grade," *op. cit.*, p. 151.

[28] Brownell, "The Evaluation of Learning in Arithmetic," *op. cit.;* Brueckner, *op. cit.*

that it can be achieved only to a limited extent if the mathematical aim is neglected. The latter aim relates to the acquisition of the content of arithmetic, to the learning of arithmetical skills and ideas (concepts, principles, generalizations, and the like).

It is not a matter of having to choose between the mathematical aim and the social aim—we must realize both aims through our teaching.[29]

Third, a systematic course in arithmetic is needed to sensitize both teachers and pupils to the mathematical elements in problems in other fields and to give pupils the confidence, resourcefulness, and competence needed to deal with these elements as they are confronted. This is one of the most important contributions that instruction in arithmetic can make.

Fourth, parents expect arithmetic to be taught and it would not be easy to convince them that arithmetic can be efficiently learned solely by incidental methods, even if it were true.

Fifth, to obtain even moderately satisfactory achievement through incidental teaching alone requires more ability than most teachers possess. Yet in determining the effectiveness of integration of arithmetic with other areas, the teacher's ability is the critical factor for, no matter how skilfully the curriculum is designed and how adequate the instructional equipment, it is the teacher who must sense and utilize the timely opportunities for integration and make adjustments on the spot to the needs of individual pupils. In the research reported on achievements in arithmetic in experience-unit plans, the teaching was presumably done by superior teachers. In some instances the teachers were *very* superior in their enthusiasm, skill, and resourcefulness. Actual results under average or below-average teachers and potential results under very superior teachers are two very different things. There appears to be a naïve assumption in some quarters that a teacher does not need a scholarly grasp of two or more fields in order to integrate them. On the contrary, integration demands broad and sound scholarship in all the fields that are to be related. The integration of arithmetic with other fields is a case in point. To make integration effective, the teacher must have a competent grasp of the unit with which the arithmetic is to be integrated, and she must be proficient in mathematics. Suppose, for example, the unit to be taught is the conservation of soil. The teacher must know the important facts about conservation, and she must have a command of arithmetic—and a more adequate command than most teachers possess. She should also be aware of the difficulties which children have in dealing with concepts far removed from their experience. She should know how the necessary arithmetical abilities and concepts are most efficiently developed.

[29] "The Second Report of the Commission on Postwar Plans," *op. cit.*, p. 200.

And even the teacher who measures up to these requirements must spend considerable time in planning the work.

No Defense of Formal Instruction

These arguments for the systematic teaching of arithmetic should not be taken as a defense for types of instruction, fairly prevalent, which consist mainly of formal, repetitive drill. As someone has said, "Systematic teaching is not synonymous with formal teaching." The members of this committee are as opposed to such methods as are the proponents of the experience curriculum. They favor the meaning theory, involving the active processes on the part of the pupils of discovering relationships, of utilizing concrete experiences, and of generalization. Indeed the meaning-and-discovery approach has been devised specifically to prevent formal teaching.[30] As compared with formal methods, the discovery-and-meaning approach has been consistently shown to result in superior achievement as well as in greater interest on the part of both teachers and pupils.[31]

Moreover, the acceptance of the proposal that arithmetic be taught systematically, with an important place in the weekly program, in no way implies a denial of the great potential contribution of other areas to arithmetic or a belittling of the importance and difficulty of the mathematical aspects of those areas. Nor does it underestimate the need of a careful plan for co-ordinating the meaningful development of arithmetical abilities in the arithmetic period with their motivation, development, and maintenance in the study of units in other fields. Maximum achievement cannot be obtained either in the arithmetic period alone or in integrated units alone. Both types of instruction are needed.

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[30] See especially chaps. iv, v, and vii of this yearbook.

[31] Carl L. Thiele, *The Contribution of Generalization to the Learning of the Addition Facts* (Teachers College Contributions to Education, No. 763. New York: Teachers College, Columbia University, 1938); Charlotte W. Junge, "Development of a Foundational Program in Arithmetic from a Mathematical Point of View" (Unpublished Doctor's Dissertation, State University of Iowa, 1944); Esther J. Swenson, "Organization and Generalization as Factors in Learning, Transfer, and Retroactive Inhibition," and G. Lester Anderson, "Quantitative Thinking as Developed under Connectionist and Field Theories of Learning," in *Learning Theory in School Situations*, pp. 9–39, 40–73 (University of Minnesota Studies in Education, No. 2. Minneapolis: University of Minnesota Press, 1949).

CHAPTER III

THE NATURE AND SEQUENCES OF LEARNING ACTIVITIES IN ARITHMETIC

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Introduction: The Arithmetic We Teach

A Way To Think

The arithmetic we teach pupils in school is a way to think about the numbers of things—about quantities, amounts, sizes. It comprises the questions, "How many?" "How much?" "What part?" and the seeking and finding of their fitting answers. By means of arithmetic as a way to think, and in the degree that we have learned this way, we (*a*) recognize number questions and (*b*) determine their answers.

Recognizing number questions and determining their answers are not separated thinking activities. They only appear so, as we consider them separately and give them different objective statements. Our knowledge of the question in a given instance suggests the answer we should seek and the way we may find it. Conversely, our understanding and mastery of the way to find answers give clues to questions, which otherwise would be obscure, and set them forth from the qualities of situations which conceal them. The number questions we recognize in situations are not obtrusive, as are the objects which strike the senses or the objective facts which claim the attention. They inhere in the situations we confront. They are under the surface of things. They do not show themselves. We must uncover them if we are to recognize them.

Thus we have occasion to move along "the way to think" that is arithmetic only in the degree that we have learned the way. If we have learned the way sufficiently well, we can recognize number questions when they are hidden in the complexities of everyday activities and in personal and social situations. But if our learning is limited, we can recognize such questions only when they appear in the simpler and less involved situations that touch our lives. We encounter no more arithmetic in the affairs of our lives than our knowledge of arithmetic permits us and leads us to meet.

Attitude and Motive

The amount of our learning of arithmetic does not merely determine the extent of our opportunity to think about the number of things. It conditions, and often determines, the way we feel about the activities of number-thinking.

If we can look under the surface of situations and discern number questions which call for answers, we find therein the stimulus and the drive for number-thinking, and we may experience the feeling of enthusiasm which accompanies such stimulation. On the other hand, if we confront but few and minor number questions, we have proportionately less enthusiasm for the business of number-thinking. Besides, a little learning leads us into complacency.

When we cannot recognize number questions except in the simpler situations of everyday affairs, we do not realize that such questions exist elsewhere. In consequence, we feel no need, and we have no need, for a better way to think in order to answer questions we do not meet. Thus, we feel no need, and we have no need, for any more arithmetic as a means of answering questions than is required to answer questions which the arithmetic we command brings to light. The way to think that is arithmetic creates its own need. In a very real sense, we do not need it until we have it. This is the reason why our attempts to provide a need for arithmetic in advance of learning are largely futile. We do not put in front what follows in rear. We do not make the trailer pull the car.

The Language of Arithmetic

To aid us to think about the numbers of things in the way that is arithmetic—actually, to make it possible for us to think this way at all—we use number names and number signs, called figures, or numerals. The names designate and pin down our thoughts as we move along in our thinking. The signs represent and record our thoughts at each step of our thinking. But these names and signs do more than designate and represent a thought at each step; they also, by the way we use them, guide our thinking on toward succeeding steps.

Number names and number signs complement and supplement each other. The former constitute the oral language of our thinking, and the latter, the written language. They do not bear, each to the other, the false distinction which we sometimes ascribe to them, namely, that we use the oral language in "concrete" arithmetic which we understand, and the written language in "abstract" arithmetic which we do not understand. We encounter no such distinctive "concrete" and "abstract" phases in our concept of arithmetic as a way to think. Thinking in arithmetic is the same kind of thinking, whether the questions we recognize

and the ways we seek answers are simple and easy or complex and difficult. In either case, the medium of our thinking is the language we use to designate and pin down our thoughts and to represent and record our thoughts. In our thinking, we use number names and number signs together and interchangeably. We may speak them and write them and do no thinking, but this is not the fault of the language itself. On the other hand, the number-thinking we can do without them is next to none.

We may speak the number names to indicate the figures we write or we may write the figures to record the names we speak, but this is only a minor use. Of itself, such activity is not thinking; hence, not strictly arithmetic. The real relation between the number names and the number signs is not a direct one but emerges indirectly by way of the number ideas or the number-thinking they express and record. Thus, each in its real use comes into relation with the other in the degree that they are used as a common medium of number-thinking.

In our number-thinking, we use single names and compounded names, and we use single signs and compounded signs. We use single names and signs for single number thoughts or ideas. We use compounded names and signs for compounded thoughts or ideas. We compound our single ideas and our single names and signs, all according to a definite pattern. The pattern is the number system. We use the system and rely on it to keep our thinking straight, to keep our ideas in order. Number names and number signs in a number system give us the vehicle to carry along our number-thinking.

The Conditions of Learning

To teach arithmetic, we teach the use of number names, number signs, and a number system. To learn arithmetic, the pupil learns the use of number names, number signs, and number system. The use is double: to stand for a thought and to move thinking along. The pupil must think a number idea in order to have one to name or write. He must think ahead for his words and signs to serve as guides. Otherwise, his words are empty and his signs without point.

To learn arithmetic, the pupil follows a set pattern of thinking. We, his teachers, do not invent the pattern. The situations of his everyday life do not dictate it. The pupil does not choose it according to his tastes or adapt it to his individual peculiarities and predilections. He adapts himself to the pattern. It is for us merely to recognize what the pattern requires and to see that the pupil recognizes and respects the requirements.

There are then two conditions of the pupil's learning: (*a*) his thinking and (*b*) the pattern he must follow. Though these conditions operate together as one, we shall discuss each in turn.

The Nature of the Learning Activities

The Pupil's Thinking

NUMBER-THINKING AS A MENTAL PROCESS

The pupil learns the way to think about the numbers of things that we call arithmetic as he explores the way. He learns the road of thinking and how to move along it as he travels the road. The speed of his movement is of minor importance as compared with the fact that, to progress, the pupil travels under his own power.

We, his teachers, do not explain and thus make clear to the pupil the way to think about the numbers of things. The way to think is not a physical process that we can exhibit. We may not set up and start the process and say, "There it is. See how it runs." Nor may we merely say about it as we may about a physical process, "No, we do not do it that way. We do it this way."

Even at its beginning—rather, especially at its beginning—we should be aware that the way to think about the numbers of things is not physical but mental, not objective but subjective. Then we may avoid the double fallacy of our pedagogy that at the outset arithmetic is "concrete" and thus may be clear, whereas later it is "abstract" and thus must be vague. We may go so far as to improve our pedagogy by deleting from its vocabulary these contrasting, hence misleading, terms.

The pupil at the beginning of his number-thinking handles objects that are "concrete" in ways that are "objective." He does this—we guide him in doing this—as the means to stimulate and control his thinking. But the way the pupil must think, must pay attention, is not at all concrete or objective. There is probably no more fallacious principle in the whole of our pedagogy of arithmetic than that the quality of "fiveness," for example, somehow emerges of itself from contacts and experiences with groups of five. The fact of the matter is that the pupil who starts to learn arithmetic carries his thinking to the group and enlarges and improves his thinking through his study of the group. It is the quality of his thinking that matters, not the qualities of the objects he handles.

The pupil's thinking at the outset gives drive and direction to his later thinking. Each phase of his thinking becomes so intimately a part of succeeding ones that the pupil is equipped to move into a new phase as he approaches it. As he moves ahead, no phase turns out to be unknown and untried, but each a mere extension of all that he has learned to do. To illustrate, when the pupil can determine the size of a group, he can determine the excess of one group over another and the results of various combinations and separations of groups; when he can add ones, he can add tens; when he can deal with parts of any size, he can deal discrimi-

nately with parts the size of *hundredths,* called per cents; and so on. Hence, our view of the procedure of the pupil's thinking may not properly be that of the pupil absorbing contrasting features, or elements, such as "concrete" arithmetic and "abstract" arithmetic, which we provide him. Instead, our view may properly be one of on-going progress by the pupil in a continuous extension of the activity of thinking.

THE PUPIL AT WORK

Our view of arithmetic in the school is a view of the pupil in action, apprehending number questions in proper sequence, doing his own thinking in his search for answers, and paying attention throughout in the way that develops the ideas he can call his own. From the start, the teacher properly does not disclose the arithmetic the pupil should learn. Instead, from the start, the pupil determines it—thinks it out—for himself.

The teacher does not say, "We have only five chairs at the table. We need three more chairs." The pupil finds how many chairs are at the table and determines how many more are needed.

The teacher does not separate a group of ten blocks into twos. The pupil counts out the ten blocks, divides them into twos, and counts the twos.

The teacher does not announce that nine sixes are fifty-four. The pupil says, and thinks as he says, "I can find nine sixes. I know that *five* sixes are thirty and *four* sixes are twenty-four. So I know that *nine* sixes are fifty-four."

The teacher does not solve the problem of "long division." The pupil thinks, "I can divide by five. I will divide by five tens the same way."

The teacher does not reveal the "hidden question" in a two-step problem and does not indicate what the pupil should do first and next. The pupil searches for the "hidden question," he finds it, and he acts as the revealed question tells him.

The teacher does not decide whether the pupil should multiply or divide by the per cent, or which of two numbers he should divide to find the relation. The pupil, knowing the distinctions between the "three kinds of problems," looks into each as he meets it to discover its distinguishing feature, and, knowing what he looks for, he knows what he finds.

At every point the pupil does the work. He arranges. He notices. He considers. He makes up his mind. As he works, so he learns. As he works, he makes what he learns his own. As he works, he knows what he learns and is sure of what he knows.

THE TEACHER AS GUIDE

For the pupil to become familiar (*a*) with the questions, "How many?" "How much?" "What part?" so he can recognize them in the complexi-

ties of his experiences, and (*b*) with the ways of determining their fitting answers so he can seek them confidently and independently, he must practice (*a*) considering questions and (*b*) finding answers.

At this point we reach what seems an impasse in our explanation: To be able (*a*) to recognize number questions in the situations that touch his life and (*b*) to determine answers, the pupil must have learned the way to think that is arithmetic; and to learn the way to think, he must (*a*) recognize number questions and (*b*) determine answers. This is much the same as saying to a tourist who needs guidance, "To travel this road, you must know it; but before you can know this road, you must travel it."

In the case of the tourist, we map the route he should take and we point it out to him. In the case of the pupil, his teacher must do the same thing. While he is yet unable to recognize number questions, his teacher must point them out and make them clear. While he is yet untrained in the art of finding answers, his teacher must make clear what the finding of answers requires. His teacher must make clear what the pupil must look for and observe as he starts each unexplored part of his intellectual journey, at one place, then another, along the road of number-thinking. Since his journey is intellectual, its nature and extent are determined by the objects of his attention and the ways he attends to them.

CLEARED QUESTIONS

To indicate number questions to the beginning pupil, his teacher must exhibit them in simple settings and freed from distractions; for, to sense the questions, the unlearned pupil must give them his full attention. His task is to pay attention to the distinctive features of each question he confronts or to the essential activities it indicates he must do to determine its answer. To illustrate, the questions, "Two and three are how many?" and "Two threes are how many?" require, in each case, attention to the single group which the smaller ones, when brought together or thought together, compose; the question, "How many more?" calls attention to the excess of one group over another; the question, "How many twos are in ten?" indicates that the pupil should consider the total, the equality of the smaller groups, and finally their number; and so on.

Learning a number question so one can recognize it later is much the same as learning to recognize a printed word. To learn the word "dog," for example, the learner must examine its distinguishing features. At the time, he cannot be distracted by the animal or its picture or the activity, or thoughts of the activity, of playing with the animal. In the case of the number question, the pupil, to be able to consider it, must have it withdrawn for the moment from all the attractive, and distracting, situations of everyday experience in which it normally is a part. Later, when the

pupil has become very familiar with the question, he can deal with it in the situations of everyday experience, for now he is able to withdraw it from its setting for any such momentary consideration as he needs to give it. At any time, at the outset or later, to deal with a number question, the pupil has to take it aside and consider it.

In presenting a number question for the pupil to consider, it makes no difference whether the teacher asks it orally or in writing, provided that in each case the pupil understands the language. Thus, the teacher may ask, "How many twos are in ten?" If the pupil understands, he is ready to give the answer, if he knows it, or he is ready to seek the answer and then give it. Or the teacher may raise the same question by writing it: $2\overline{)10}$. If the pupil understands, he is ready to write the answer, if he knows it, or he is ready to seek the answer and then write it. Difference in the language, as we have pointed out, makes no difference in the kind of thinking the pupil should do. The language does not make the work "concrete" in the one case or "abstract" in the other. The thing of importance is that the pupil should know what a question asks, whether he hears it spoken or sees it written.

SEEKING AND FINDING ANSWERS

To make clear the way to determine answers, the teacher must do more than exhibit a performance for the pupil to imitate. In his arithmetic, if anywhere in his school work, mere imitation is not conducive to intelligent response. Here again, it is the way the pupil attends and the object of his attention that count. So, instead of merely showing the pupil what to do and how to do it as a guide to his imitation, the teacher must guide him to seek the answers himself. For the pupil to do this, he must have an *inner* guide, as it were; he must know in every instance the *kind* of answer he should seek. Then, and only then, will a suggestion or a demonstration be a real aid.

To search for a book, for example, a person needs more guidance and incentive than the general idea, "find the book," can give. He needs to have in mind in advance, and while he is searching, what particular book or kind of book he should seek and find. Similarly, in his search for the answer to a number question, the pupil should have in mind in advance, and while he is searching, the particular kind of answer he should seek and find. Thus, the pupil should know, if his question is "eight and seven," that his answer must be a *teen;* if his question is "four nines," that his answer must be *tens;* if his question is, "What part?" whether he should find it in terms of nearest *tenth,* or *hundredth,* or *thousandth* part; and so on.

Knowing the kind of answer he should seek aids the pupil in several

ways. For one thing, it is the best possible check he may have at his command on the accuracy of his work. For another, it is useful in determining the procedure of his calculations. When he divides by decimals, for example, he confronts the issue of whether to use the caret in order to deal with the divisor as though it were a whole number. The issue resolves and disappears, if his thinking is dominated, first and last, by the *kind* of answer he should seek, whether in nearest *tenth,* or *hundredth,* or *thousandth.*

Knowing the kind of answer he should seek aids the pupil in yet another way, and at the same time relieves a serious doubt which frequently disturbs his teacher. When the pupil starts his arithmetic by doing his own thinking as we have indicated, his actions are clumsy. He must crawl, as it were, before he can walk and must take faltering steps before he can run. In short, his methods of learning have but a faint resemblance to the methods of use which his teacher wishes him eventually to pursue. Will he not form habits of awkward, roundabout ways of thinking which will slow down and render ineffective his later work? What must the teacher do to prevent the earlier methods from cluttering up the methods the pupil needs to pursue later? Such questions should not disturb the teacher who makes sure that the pupil knows the kind of answer to seek. The questions disappear before the independent work which the pupil learns to carry on. The methods of work the pupil uses in any instance adjust themselves, and the pupil adjusts them, to the kind of answer he has in mind to seek. At no point do his methods dominate his thinking, for at every point his idea of kind of answer dominates his methods. In this way, his methods of learning stay fluid. As his idea improves, his methods improve. They develop slowly but surely from clumsy methods of learning into effective methods of use.

PROVIDING FOR THE OUTCOMES

The pupil at work doing his thinking under the stimulation and guidance of a developing idea of what he should accomplish resolves other doubts, or questions, which commonly disturb his teacher. His teacher interests himself in results—skill, meaning, favorable attitude, independence in interpreting, practical applications—and in getting them in workable balance. Will the pupil become skilful in his calculating and at the same time gain the necessary meanings for intelligent use? Will he transfer and apply his computations properly to the solution of problems? How may he bring all the results of his training into suitable relations?

The teacher finds answers to all such questions when he considers results less in terms of goals for the pupil to attain and more in terms of the quality of the pupil's thinking. Skill in number-thinking is not reflected by speed, but by deliberation; it is not quick reflexes, but controlled re-

sponse. The pupil improves his skill through the practice of number-thinking. In similar manner—through practice in number-thinking—he expands what arithmetic means to him and enlarges his view of the opportunities to use his number-thinking. What the pupil learns who learns arithmetic is not a set of qualities which the teacher makes static by names, but rather a doing of something, the paying attention to certain things in certain ways. The teacher makes it difficult to think about number-thinking by using too many nouns that pin down and not enough verbs of action.

To illustrate, let us translate the matter of meaning and application as outcomes into the actions of thinking by the pupil. Let us begin by raising the question, "How does the pupil acquire the meaning of *division*, for example?" But that is not the way to ask the question. Such a question suggests that dividing is one thing learned by the pupil; the meaning, another. The way to ask the question is rather: "What does the *pupil's* dividing—the dividing that *he* does—mean to him?" If we look, we see that dividing means just what he does and thinks as he divides, no more and no less.

He divides to answer a question: "How many twos are in eight?" or "What is one-half of eight?" He separates his original group into twos or into two equal groups, and he counts the groups or the number in each, as the case may be. If the question is clear (this is a matter for the teacher; the teacher must make the question clear), and, if the pupil preseveres (something else the teacher may look after), he pays attention, as he divides, to the equality of his groups or parts and to the number of them or within them, as the case may be. Thus, he comes to the answer, *his* answer, because he found it by *his* work and thinking.

This activity and its succeeding and expanding repetitions constitute the first step in the pupil's thinking and learning about dividing, in the case of the present illustration. The question for which he seeks an answer, the kind of answer he should seek, the dividing he does, and the objects of his attention throughout, are all in the simplest possible setting. Such a setting has its concrete or objective features reduced to the necessary minimum.

Here, within a single activity of the pupil are all our separately named outcomes as one: *independence*—he does his own work; *attitude*—he attends to equality and number as he works and, at the end, the answer found being his own, he takes pride in his achievement; *skill*—this is the way he works and thinks, the way he has learned by working and thinking; *meaning*—he knows the kind of answer he seeks and the answer he finds; *application*—this is merely a continuation of his thinking, of his recognizing of number questions, in the complexities of the situations which he learns to meet.

THE SINGLE ROAD OF NUMBER-THINKING

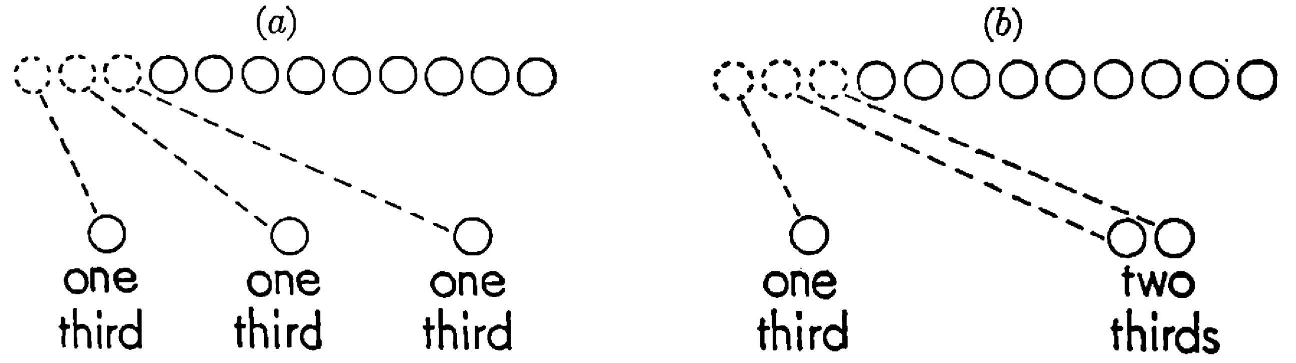
Let us follow the pupil who finds his own answers to division questions that are clear. As his *first* step in thinking and learning, he divides, attending to the equality of his groups or parts and to their number or the number within them. He so works, or studies, or thinks, both when he divides actual groups and when he merely thinks his dividing.

As his *second* step in thinking and learning, the pupil so continues when a practical situation presents his division question. The only difference here is that he divides and considers equality and number amid the distractions of the practical situation. The practical situation adds to the complexity. It adds also, if the pupil is ready, to his training in dividing.

Finally, if the pupil keeps on, he becomes so adept in thinking his dividing that, as a *third* step in thinking and learning, he thinks the essentials of dividing in connection with a practical situation which heretofore has been unfamiliar. He studies the new situation as a kind of special case of his dividing. In this way, he turns the new situation into a familiar one.

To illustrate, we may follow a pupil in his dividing of a group of 12 objects into thirds, for example:

First step. At the outset, the pupil may experiment (*a*) by taking objects from the original group of 12 and arranging them in 3 equal sets, placing the objects in these sets at three separated spots on his desk. Thus, he separates his group into *thirds*. Or, (*b*) giving special attention to a *single* third which he seeks, he may place one of each set of 3 at one spot and the other two at a second spot. Thus he finds *one* third of his original group.



Later, he learns to substitute the familiar method of calculating for the original procedure:

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   4          25
3)12        3)75
  12          6
              15
              15
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Second step. Now the pupil deals with "problems": "Last week Helen earned 75¢ baby-sitting. She put one-third of the amount she earned in the School Savings Bank. How much did Helen put in the School Savings Bank?" In such "problems," the situations are familiar.

Third step. The pupil carries his dividing to situations that require study: "Jack and Don had a lemonade stand. Jack furnished one-third of the materials. The profit, after they had paid all expenses, was 90¢. How much was Jack's share?" In this "problem," the pupil has to become clear on the meaning of "profit, after expenses." He must be clear that proper "wages" must be allowed by the boys as part of their expenses, in order to consider Jack's share as one-third of the profit.

THE NUMBER CHARACTER OF THE PRACTICAL SITUATION

The practical situation, and the number relations that we commonly describe as "involved" in it, are but a single item of experience. True, we can abstract, or take out, the number relations and view them alone. But we cannot view the practical situation with number relations abstracted, because then there is no practical situation. In fact, the number relations of a situation are what make it a situation. This is the reason why there is really no such thing as social arithmetic per se, however active our minds as we imagine it.

The number relations that a person learns affect him personally; hence, they determine his attitude. They may, at the same time, affect another person or many others. Now they are social in influence. A transaction of buying and selling has its personal and its social sides. It may please one or all parties concerned, or it may not. Apparently, the act of buying and selling is one thing, and the number relations "involved" are another. So we imagine, but it is contrary to the fact. The act of buying and selling does not exist except in terms of the worth of an article, considered in number of dollars, and of the number of dollars given and received in exchange. According to the way these numbers of dollars balance is the sale pleasing or displeasing. The well-known fact is: no dollars—no sale.

Again, taxation is a matter of politics; it affects business; it is a social and personal concern. To it, we apply our arithmetic, such as we have, when we need to or when we feel like it. Again, not so. A tax is a tax, good or bad, effective or a nuisance, only according to the numbers of dollars it assesses on numbers of persons. Suppose we try to imagine a tax that omits mention of rates. We have somewhat the same result as the statute enacted by a legislature which defined a certain act as a crime but failed to fix the penalty. We could go on naming all the practical situations in which we erroneously think we *apply* our arithmetic—rent, interest, insurance, commission, and so on—and in each and every instance we have no situation except such as the numbers and number relations, that are its warp and woof, make it a situation.

In short, the pupil does not first learn his arithmetic and then apply it, as separate items of experience; he does not develop his skill, get his meaning, acquire his attitude, make his application, and become inde-

pendent, either at separate times or along separated pathways at the same time. He does them all at once, or rather as one act, as he moves along the single road of number-thinking.

The Place of Problem-solving

NUMBER-THINKING AND SITUATIONS

The quantities, amounts, and sizes of things which the pupil should learn how to think about are quantities, amounts, and sizes of any and all things, whether imagined or actual, unimportant or important. The pupil realizes this fact as he learns the way to think about them. To illustrate, when and as he learns to count at the very outset of his learning of the way, he is aware that he can count anything; and this he does. He counts the chairs in the room, the books on the table, the pictures on the wall, and the trees in the yard. Yet we, his teachers, are inclined to ignore the evidence before our eyes. We consider the mere reciting of the number names as counting of a sort—"rote counting" we call it—and then we consider, as something separate and different, the possibility of giving counting a practical use. We forget that "count" is a transitive verb and that it indicates no action when apart from its object.

Similarly, we tend to set apart as different kinds of activity subsequent phases of the number-thinking the pupil learns to carry on and their "uses" or "applications." We seem to feel that number-thinking which can be about anything in general is necessarily about nothing in particular, hence, without meaning until we carry it to a situation which gives it meaning. Thus we lead ourselves into error and confuse the pupil, as we have already noted. Thus, moreover, we perpetuate a cleavage in arithmetic which we inherit from tradition.

Computation and problem-solving grew up and developed as separate interests of the arithmeticians of the past. Each has a history that differs from the history of the other, and each has moved out of the past into our present school practices as a separate and distinct activity. Our predecessors seemed more impressed by the external differences of the activities than by their internal similarities, by the way they appear than by what they actually are. So do we. We continue to separate the activities in our books, our courses, our methods, and our testing programs.

It is true, of course, that different situations present different difficulties, sometimes different problems; but such differences do not change the character of the number thinking by which we interpret the situations. We must put forth different degrees of effort and use quite different methods to resolve costs, capacities, weights, heats, energies, and so on, into countable units; but, when we so resolve them, we count the various sets of units. Our modes of counting and our combinations and separa-

tions do not differ from those of the pupil, except perhaps in degree of maturity. We have no reason for the view that a given phase of number-thinking is made different in varying situations. We have abundant reason for the view that number-thinking is number-thinking wherever we may have the pupil engage in it.

Yet there are many differences in the situations in connection with which the pupil may practice the number-thinking he is learning. There are simple situations and complex ones, familiar and unfamiliar situations, important and unimportant situations. We must choose the situations with which the pupil should deal. We need a set of values by which we may judge the situations we should choose.

NUMBER-THINKING NOT A PROBLEM

The phrase "problems in number-thinking" is a contradiction in terms. A problem is a question involving doubt. About the number-thinking the pupil learns to do there is no doubt. It is *his* thinking, and about it he can be sure. It is only when the pupil has not learned the way to think that is arithmetic, or when he has learned it imperfectly, that he is uncertain. The characteristic quality of a person's number-thinking—the thinking *he* does—is its certainty, its reliability.

The problem—the doubt that exists—is in the situation that is unfamiliar, in the situation not fully sensed, or improperly viewed, or inadequately described. The problem is not in the number-thinking, provided that the pupil has already learned to engage in it.

If we require the pupil to deal with an unfamiliar situation, we give him a problem. If we cause him to study the situation, we help him to make it less and less a problem. All such exercises of study, including exercises with familiar situations about which there is no doubt, go by the name "problems." For convenience, we so use the term in our discussion.

THE LOCATIONS OF PROBLEM EXERCISES

There are three positions in which we may place problem exercises for the pupil to consider: (*a*) at the beginning of his study of a new number process or phase of number-thinking, (*b*) in the midst of his study, and (*c*) at or near the end. The problem exercises at the three points differ in kind. They present situations for three different purposes: (*a*) to provide a motive for the study of the new number process or phase of number-thinking; (*b*) to contribute to the study by extending the pupil's practice in thinking; and (*c*) to give training in using number-thinking in the study of situations. We have already touched upon the "steps" of thinking and learning which parallel the three purposes.

A MOTIVE FOR LEARNING

In the preceding section, we referred to the beginning stages of the activity of dividing into equal groups and equal parts to illustrate what the

pupil's dividing means to him. There we indicated that the teacher should start the pupil with the question, "How many twos are in eight?" We indicated that the teacher made the question clear, thus making clear the kind of answer the pupil should seek, and that the pupil proceeded to count out eight objects, to separate them into twos, and to count the twos, thus arriving finally at his own answer to the teacher's question.

When we read such a statement of the initial steps of the pupil in dividing, we, his teachers, are inclined to halt and raise the questions, "But what reason does the pupil have for dividing eight?" "Why should he wish to find the number of twos?" "Why should he care if he does not know what good the information will do him?" "Shall we start the pupil's dividing, as it were, in a vacuum?" Thereby we indicate a lack of understanding both of the pupil and of the arithmetic we should wish him to learn.

These are questions *we* raise, never the pupil, unless we in error have so taught him and have put the words into his mouth. We are interested in the long sweep of the pupil's movement through the course in arithmetic, not merely in his activities of the moment; we look toward ultimate ends as well as toward immediate ones. The pupil, on the other hand, does not see the matter from our mature point of view. Much depends, therefore, upon the sources of our answers, whether they come from the fruits of our experience or the pupil's. Thus, if we can take the pupil's view when we raise the questions mentioned, we may find that the resulting answers are far from being unsatisfactory.

THE PUPIL'S MOTIVE

The pupil, provided that he has been doing his own thinking and has moved ahead to the point where he can consider dividing, has reasons and wishes and attitudes that are favorable for the new activity. His *reason* for dividing is that he recognizes things he knows he can do and something slightly new and different to attempt. He *wishes* to find the twos in eight because his teacher has made the question clear, and the clear question suggests a way to find the answer without actually revealing it. He *cares* about the information which he may get because the getting gives the promise of personal achievement.

Measured by our standards, these answers seem small indeed; but measured by the pupil's standards, they assume large proportions. The pupil, who has been thinking his answers to the point where he encounters the new question that requires his dividing, has gained a momentum, a drive in his thinking, that does not suddenly stop. He has a motive for driving ahead that is supplied by what he has done and by what the new question suggests that he possibly can do. The new activity invites a trial. Its already familiar phases give confidence. The answer, as yet un-

known, excites curiosity. His past success inspires self-dependence. So the pupil does have a motive for dividing, the logical motive of his on-going progress in thinking. He does not have the mature practical motive which we have gained. We do him a disservice when we try to supplant his motive with ours. We fail to supply our motive and we may destroy his.

MAKING THE QUESTION CLEAR

Yet we feel that we must pursue our questions: "Must the pupil do his work in a vacuum?" "Why should we not supply a practical motive that is practical to him?" Thus, instead of asking the "bare" and "barren" question, "How many twos are in eight?" let us ask the kind of question that has some slight practical bearing:

Helen is making sandwiches for a party. She has 8 slices of bread. How many sandwiches can Helen make?

Helen needs 8 buttons for her doll's dress. The buttons come on cards, 2 buttons on each card. How many cards of buttons does Helen need?

We certainly should not wish our pupil to work in a vacuum or to deal with the dull and the drab. We should concede that the problem approach, as indicated, may give his work substance and add to the personal motive for undertaking it. We should note the possibility that the use of problems may make clear a phase of the work to which the pupil must attend. In the present instance, the "two slices" to a sandwich in the one problem or the "two buttons" in the other may serve to emphasize the *equality* of the smaller groups that should result from the dividing to be done. Yet we should use the problem approach with care and caution.

In the present instance, whatever the help from the problems, the pupil must start his dividing with attention on groups of two, not on sandwiches or cards of buttons. To him, the sandwich or the card may be a poor substitute for a group, though to us either seems a clear representation. To start his thinking, he usually must abandon the apparently clear representation in order to produce his groups for himself.

Besides, we should expect the pupil's answer, even his first answer, to be a generalized one. The question, "How many twos are in eight?" though it requires the pupil to deal with an actual group of eight, does not bear upon any particular kind of objects. Though the pupil needs eight objects, he may use any objects at hand—blocks, discs, books, or sticks. The particular objects do not take his attention, but the way he arranges them; hence, his answer is a generalized answer which applies to any like division of eight. On the other hand, the two problems raise questions about particular things—sandwiches and cards of buttons—not things in general. The answers resulting are particularized answers.

EXTENDING THE PRACTICE

When the pupil generalizes the answers that result from his dividing—or finding the total, or average, or number of hundredths—and has gained some facility and confidence through practice in formulating generalized answers, he is ready to particularize them. He is ready for "problems" which raise questions about particular things, such as sandwiches and cards of buttons, as in the illustrations already given. And he is ready to deal with the partly familiar questions when they are somewhat obscured by the concrete features of particular things and experiences. The less obvious and slightly hidden questions require closer attention than do those with which he deals at the outset and, in consequence, they extend his practice in thinking.

To be suitable for practice, the problems must have two features in addition to "statement in simple and direct terms." They must deal with situations that are both familiar and varied. Their purpose is to give practice in recognizing questions about equal groups or parts—or about totals, or averages, or per cents, as the case may be. To provide opportunity for practice, their situations must be familiar so as not to hide completely the questions the pupil should recognize. Thus, for example, if the problem inquires about the shares of *A* and *B* in the profits of an enterprise, it makes a difference whether the two persons are equal partners or have made unequal investments. Either situation determines the nature of its question. In the case of either, the pupil must be perfectly clear about the nature of the situation.

To increase the pupil's familiarity with the question we wish him to consider, whether about equal groups or parts or about something else, he must encounter the question *and recognize it* in varied guises and circumstances. Each occurrence reveals the question in different form and reveals it from a different angle. Each contributes to the pupil's knowledge of the question, whatever its form. Finally, through repeated recognitions, the pupil may become so familiar with the question that he is able to recognize it in *any* form or circumstance, even when disguised by the distractions of new and somewhat unfamiliar situations.

STUDYING NEW SITUATIONS

To become acquainted with the new and unfamiliar situations which we commonly call "social," the pupil has to study them. Each is a fluid complexity of many elements and each assumes many different forms. They do not fit into a given mold, except such as a common mode of number-thinking may provide; and they do not conform predictably to certain rules. To learn about them, the pupil has to take them apart and give them a critical view.

To be able to study social situations, the pupil must be prepared. He must know in advance the common question each of several socially dissimilar situations poses. The common question, requiring a mode of number-thinking which the pupil has learned to carry on, is their central feature, and it provides a common way of proceeding toward the appropriate answer. To illustrate, spending money and losing money are two very different situations, yet the idea of subtracting leads to a common and clear result. To illustrate further, the payment of interest, the receipt of a profit, the allowance of a discount, the charge of a commission, and the levy of a tax are all different, yet they all are characterized by the common question that indicates each result in the terms of so many *hundredth* parts. The pupil who approaches the study of all such problem situations with a clear idea of *hundredth* parts has the single key which unlocks the answers to all. He is able to bring all such different situations into the single category which a common mode of number-thinking provides.

Social situations are *social* according to the *number* of persons they involve. They are *situations* because in them the *number* of persons involved, the *number* of activities employed, the *number* of dollars expended, and the *numbers* of other factors that may be present appear in unpredictable, fluid, and evasive relations. They are problem situations to those who are able to sense the challenge they offer to bring their elements into balance. The pupil can apprehend social situations and make them objects of study only in the degree that he has become aware of the number relations which characterize them.

Social situations do not supply meanings to number relations, at least not directly. If anything, to continue the manner of speaking, exactly the reverse is true. Social situations provide a medium in which the pupil may relate numbers and thus work out what the relations mean to him, as we have indicated as *step two* (see pp. 31–32). Yet when they are not clear, it is what the pupil has determined the number relations to mean to him that enables him to study the social situations and to find out thereby what they do mean. To illustrate, considering the features of installment buying does not make percentage clear to the pupil. Unless the pupil already can deal with *hundredth* parts and the relations they show, he is unable to consider anything about installment buying except its surface, objective features. Like the credulous customer, the pupil may consider the conveniences of installment buying and their cost as charged, but the cost as a relational amount does not reveal itself. The pupil must become as the wise customer by developing the idea of relational amounts: (*a*) He gets this idea by dealing with relational amounts in simple and untramelled form. (*b*) He develops the idea by repeated recognitions in familiar situations. (*c*) He, finally, may use the idea, as in the present instance, to make

clear the cost of installment buying. He now can gauge such cost in its true proportions.

THE COMMON QUESTION

We have indicated three steps of progress in the pupil's practice in thinking. Each distinguishes itself as a kind of special way for the pupil to deal with a common question. In step *one*, he deals with the question in its simplest form and with the fewest possible adornments. In step *two*, he deals with the question in forms that are made complex by the concrete features of familiar situations. In step *three*, the pupil makes a new departure. He deals with the question as a key to open to his view the inner features of complex situations which otherwise remain hidden and unfamiliar.

THE SEQUENCES OF THE LEARNING ACTIVITIES

The Pattern of Thinking

THE SOURCE OF THE PATTERN

The arithmetic we teach the pupil provides the pattern of his thinking. It is, as we have said, a way to think about the numbers of things. To learn the way, the pupil must travel the way. To travel the way, he must guide his movement of thinking as the course of the way requires.

The arithmetic we teach the pupil is the arithmetic of the Hindu-Arabic numerals. This arithmetic treats the groups of various sizes to ten as single groups. It breaks down groups larger than ten into teens and tens and treats each such group according to its relation to ten, whether a multiple of ten or a power of ten. This arithmetic is a system, a system of tens.

To learn this arithmetic, to learn to *use* this arithmetic, the pupil must learn to accommodate his thinking to its requirements. He does not choose the mode of his thinking. This arithmetic dictates it. The pupil is successful in so far as he recognizes and performs the acts of thinking which this arithmetic requires.

WHAT ARITHMETIC REQUIRES

a) Arithmetic requires the pupil to develop ideas of the single groups of each size up to ten, to give each its single name, and to represent each size up to nine by its single numeral.

b) Arithmetic requires the pupil to study the standard group of ten, including the teens and tens, and to learn the special way of writing ten and its relations.

c) Arithmetic requires the pupil to learn as teens and tens the combinations that exceed ten.

d) Arithmetic requires the pupil to deal with tens and the powers of ten the way he deals with ones, that is, with the single groups to nine.

e) Arithmetic requires the pupil to study the sizes of parts, to represent them according to their uses, and to combine related parts.

f) Arithmetic requires the pupil to carry his study of the fraction to the point where he can understand and use it as a way to express the relations between quantities.

g) Arithmetic requires the pupil to use his ideas of relational amounts as means to familiarize himself with the nature, import, and consequences of the various social, business, and civic situations and transactions that exist and go on around him.

h) Arithmetic requires the pupil to determine amounts in terms of the standards by which, and in respect to which, we measure them.

DETERMINING WHAT NUMBERS MEAN

The pupil lives in a world in which the vital questions about every situation are "How many?" "How much?" "How big?" He lives in a world of number ideas—quantities, amounts, sizes. To become an intelligent citizen, he has to learn, among other things, to interpret numbers, small and large. He has to master the technique of interpretation.

Mastery of a technique does not come through casual observation. The boy does not gain expertness with his bicycle by riding it once around the block or by using it occasionally to run an errand to the store. Neither does the pupil master the art of number-thinking through an occasional, desultory, practical application of this or that special process. Continuous and progressive study of numbers, small numbers and large numbers, is required. The pupil must face, he must be brought to face, the requirements of learning arithmetic. He must observe the requirements and respect them. His study runs through four somewhat overlapping stages.

STUDYING WHOLES

At the outset, the pupil learns small numbers by studying groups to ten. He then learns the relations that somewhat larger groups bear to ten as the base, and he learns and practices systematic combinations and separations in relation to ten as the base. Practice gives skill which has little immediate service. Thoughtful practice, and much of it, gives a widening mastery of larger and larger numbers. Finally, a limit to the advantages of even thoughtful practice of computations is reached. At this point, an enlarged method of studying numbers must be found.

STUDYING PARTS

The pupil learns the enlarged method through an extended study of parts. At the outset, he studies parts as various-sized divisions of wholes, paying attention to the common sizes as they relate to the whole and as they relate to each other. Next, still attending to sizes, he combines and separates fractional parts. Throughout, to keep his thinking clear and

accurate, he constantly returns to the direct observation of actual parts of things and he relates his thinking to partial amounts in familiar situations. Thus, from the outset, he gives his study a practical bearing. Yet his study of the relations of parts does not stop with obvious so-called "practical" uses in everyday situations. Such are, instead, points of departure. Under our guidance, he continues to study the relations of parts far beyond the requirements of such obvious usage. We have him study parts until he gains the idea of using them to think and to express the relations between numbers. Now, he has a new way to study and learn about numbers and what they mean, namely, the fraction as a relational idea.

STUDYING RELATIONS

The pupil develops the new idea through practice in dealing with what we have come to call "the three kinds of problems": finding the part, or per cent, of a number; finding the part, or per cent, one number is of another; finding a number when a part, or per cent, of it is known. He uses the idea as a kind of common triple-headed question to bring into order and under classification many varied situations that touch his everyday life and with which he should become familiar. Through practice in finding the answers that are called for in particular instances, the pupil learns to consider and to value as relational amounts the numbers he seeks to study and interpret. In short, he studies numbers—quantities, amounts, sizes—through computations which combine and separate, as at the outset, and now bring out comparisons. Thus he makes numbers, small and large, mean more and more to him.

INTERPRETING NUMBERS

The arithmetic the pupil learns does not confine itself to a single subject in his curriculum. It extends to, runs through, and colors and flavors every other subject. Though his systematic study of numbers may be confined to the organized activities of a single class period, his interpretation of numbers has an active and determining part in all others of his learning activities (see chap. ii).

In his other classes the pupil must consider the qualities, purposes, and influences of things, of conditions, of happenings, and of situations about which he seeks to become informed. We may say that this is the character of his work and study outside the class in arithmetic. Yet this is but a phase. Another phase is number—quantities as well as qualities. Or if we regard the work in other classes as the single phase, yet, as we have earlier made clear, the quantities that are involved condition and determine the qualities we wish the pupil to learn. For instance, "Is a person industrious? *How many* words does he look up, or *how many* bricks does he lay?" "Is a rule a good one? *How many* persons does it aid, or incon-

venience, as the case may be?" "Is a machine efficient? *How many* are its r.p.m. or the pounds it will lift?" "Is a situation helpful? *How many* does it help?" "Is the land fertile? *How many* bushels does it produce?" In everything the pupil studies and will need to study, in school and out, the quantities of things involved constitute the determining element.

His social studies grow in importance as the pupil moves into the later stages of his work in school. Here, if anywhere, he does not need paper and pencil for computations. Still, it is *numbers* of persons, *numbers* of dollars, *numbers* of items of output, and *numbers* of everything else in every topic of his social studies that make it a topic of enough social value to study. The pupil gets nowhere in his social studies, except perhaps in ungrounded intentions of ill-defined proportions, unless he can interpret the numbers that are involved. Without this ability or without its constant use, the pupil gets only a distorted picture of the events and conditions he tries to study. Frequently, the distorted picture is more harmful than no picture at all.

The need to interpret numbers is not confined to the later studies of the pupil. His history confuses him until he can set its events into periods which are marked off by *numbers*. So does his geography until he considers *numbers* of miles of distance, *numbers* of persons, *numbers* of activities, and *numbers* of products. So does his course in health and safety. The pupil who observed, though inadequately, that "the *average* is as close to the bottom as to the top," could understand why an *average* speed of 30 miles an hour may be unsafe, why it may be inadvisable to wade in a stream that *averages* only two feet in depth, as well as why an *average* grade may be far from satisfactory. The pupil who read the figure giving the number of bushels of apples raised in a good year, then related it to the "apple a day" rhyme, and proceeded to find if there were enough apples to provide one a day for everyone, came out in the end with much more than a vaguely understood figure and a catchy rhyme. Coloring and determining all phases of the pupil's work are numbers, small and large, which, for his work to be clear, he must interpret.

RESULTING ATTITUDE

The pattern of the pupil's thinking is reflected in his attitude. There is truth in the assertion that much depends on the pupil's favorable attitude. Yet favorable attitude must, first of all, be a result before it can be a cause. In his arithmetic the pupil may learn to like what he does, which, as end product, is doing what he likes. Our frequent effort to put the end product at the beginning distorts our view and confuses the pupil. Favorable attitude is really an accompaniment, a quality, of the number-thinking the pupil does. Number-thinking, requiring alertness and respect for

its consistent rules, produces ideas. These are the pupil's own. Toward such as these he is favorably inclined.

Number-thinking gives the pupil a large degree of personal satisfaction. Mastering the art is a personal achievement, a triumph that is not confined to the brighter pupils. The slower-learning pupil finds in number-thinking something he can do unaided and alone. Mastery of the art helps the pupil to avoid the frustrations that are met by those whose number ideas are meager and confusing. Mastery carries with it a sense of independence and self-reliance. The art of number-thinking, when the pupil has gained it, is for him a personal possession which he finds dependable and valuable.

The pattern of the pupil's thinking in the learning of arithmetic is shaped by the activities which arithmetic requires of him. We have indicated these activities in their gross proportions and have noted the major stages of the pupil's progress which the activities produce. We may now indicate the sequences of the steps he takes to move ahead in his learning and thinking.

The Steps in Learning

STUDYING WHOLES

a) Arithmetic requires the pupil to develop ideas of the single groups of each size up to ten, to give each its single name, and to represent each up to nine by its single numeral. This means that the pupil begins his work by studying groups. The systematic character of arithmetic suggests that the pupil should study the groups by systematic methods. The teacher guides the pupil in the use of systematic methods. The pupil uses each method as he learns it to study the groups, and he learns each as he uses it. He studies a group

(1) by *counting* it; later

(2) by *comparing* it with another; and later

(3) by *separating* it into parts, and by *combining* the parts.

From the outset, the teacher does not tell results. The teacher asks the question, and the pupil tells the answer. The pupil tells the answer if he knows the answer. If he does not know the answer, he finds it by using the method the teacher has indicated. Now, he tells the answer. Thus, the teacher does not say, "Here are four apples." Instead, the teacher inquires, "How many apples are here?" The pupil determines, and he tells.

The pupil tells in two ways, as required. He says the *name* that "tells how many"—"four"—or he writes the *figure* that "tells how many"—4. Thus he learns to associate the *name* and the *figure* each directly with a given group he has considered, and not directly with each other.

In comparisons, the pupil studies a group in relation to another. Again, he tells the answer. "How many pupils are around the table?" "How

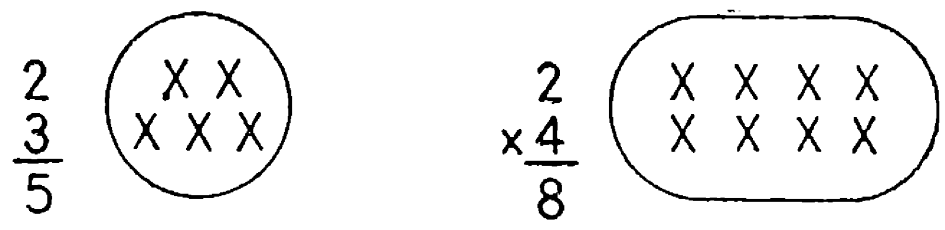
many chairs are at the table?" "How many pupils have chairs?" "How many more do we need?" Here, the pupil deals with the excess of one group over another. He determines the excess through the now familiar method of counting.

Finally, the pupil studies groups through separating and combining. Here, again, the teacher does not tell answers or give "facts." The teacher inquires, "Two from five is how many?" The pupil counts out five. He takes two away. The question now is, "How many are left?" The pupil finds and tells the answer. The next question is, "Two and three are how many?" The pupil counts the separated groups together and finds his answer.

Concurrently with the oral questions and answers, the teacher writes the question and the pupil finds and writes the answer. Thus the expression, $\begin{array}{r}5\\ \underline{-2}\end{array}$, inquires, "Two from five?" and the expression, $\begin{array}{r}2\\ \underline{3}\end{array}$, inquires, "Two and three?" The pupil thinks each answer, as above, and writes, or says and writes, each answer.

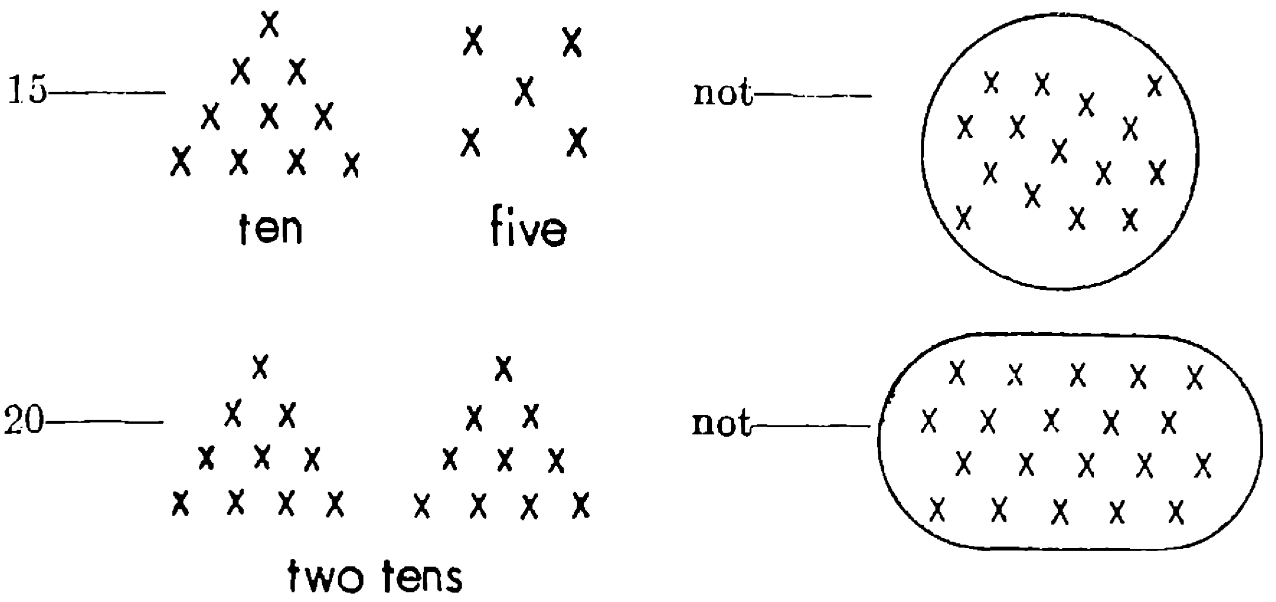
Similarly, the pupil finds and tells, or tells and writes, the answer to each of the 45 subtractions having minuends to ten, and to each of the corresponding 45 additions. Similarly, as we have indicated earlier, the pupil deals with and answers questions which require separations into equal groups: "How many twos are in eight?" $2\overline{)8}$, the teacher inquires. The pupil finds the answer, tells it, and writes it: "Four twos are eight," $\begin{array}{r}4\\ 2\overline{)8}\\ \underline{8}\end{array}$. Thus he deals with division questions involving 8 and determines their multiplication answers.

In his combining, the pupil counts separate groups into single groups: two and three into a five, four twos into an eight, and so on.



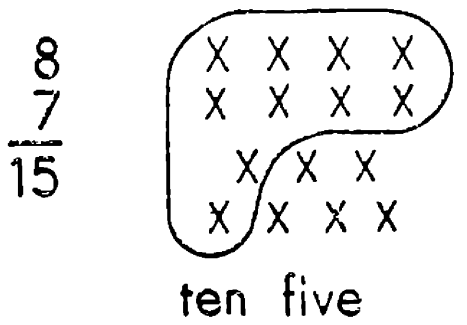
b) Arithmetic requires the pupil to study the standard group of ten, including the teens and tens, and to learn the special way of writing ten and its relations. The pupil studies again the group of ten the way he has studied the other single groups, paying special attention to the different way he writes the figures to express the idea: with the zero to aid, he writes the numeral 1 in ten's place. He thinks of each teen, not as one group, but as two groups: fifteen as a five and a ten, and so on. Similarly, he represents each, whether objectively or symbolically. Fifteen objects in two groups,

a five and a ten, either actual or as thought, not in a single mass or pile, demonstrates the idea of this teen. The figure 5, standing for one group, and the figure 1 *in ten's place*, standing for the other, represent the idea. Similarly, the pupil thinks and deals with the tens. He learns to use the word *twenty*, for example, to express the idea "two tens." He sets up two groups of ten to demonstrate the idea. He writes 2 in ten's place, using zero to aid, to represent the idea.

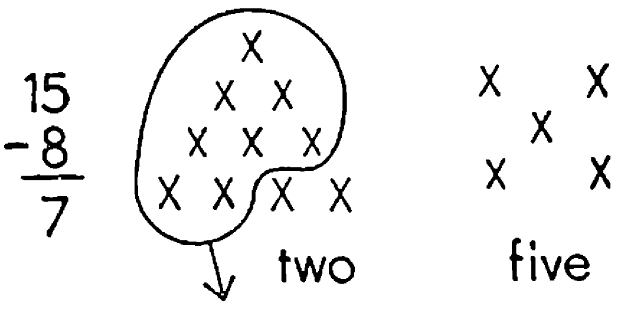


c) *Arithmetic requires the pupil to learn as teens and tens the combinations that exceed ten.* Here, as elsewhere, the teacher asks the question and the pupil determines the answer. The teacher may ask the question orally or may write it; the pupil, once he finds the answer, may tell it or write it, as required. In any case, the teacher asks the question so as to make clear the *kind of answer* that is required.

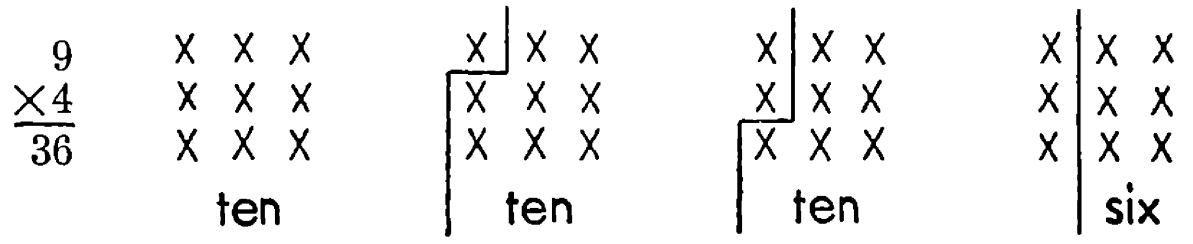
To learn "eight and seven," the pupil puts the two groups together, actually and in thought, not into a single group, but into a ten and a five. At the outset the teacher does not ask, "Eight and seven are how many?" The pupil could answer, "A dozen and three." This answer, though correct, is not the answer desired. The answer desired is a ten and a five. So the teacher asks, "Eight and seven are *ten* and how many?" The pupil works and thinks, "Eight and two are *ten*." "Two from seven is *five*." "Eight and seven are fifteen."



Similarly, the pupil may find for himself the answers to the rest of the 36 additions up to nine and nine. In reverse, the pupil finds the answer to the 36 corresponding subtractions. In finding each of these, he subtracts from a teen. He works and thinks, in the "eight from fifteen" question, for example, "Eight from *ten* is two." "Two and *five* are seven." "Eight from fifteen is seven."



To learn "four nines," the pupil puts the four groups together, actually and in thought, not into a single group, but into three tens and six. At the outset the teacher does not ask, "Four nines are how many?" The pupil could answer, "Three dozen," or "Six sixes." These answers are correct, but neither is the answer desired. The answer desired is three tens and a six. So the teacher asks, "Four nines are how many tens?" The pupil works and thinks the nines into tens: three tens and six, or thirty-six. Similarly, he may find the answers to the rest of the multiplications that remain to be learned. As he learns each, he finds that his answer to the multiplication question also fits the corresponding division question, "How many nines are in thirty-six?" $9\overline{)36}$.



The pupil does not need to find answers to all his additions and subtractions in the teens and to all his later multiplications by the method we have indicated. He needs to find sufficient answers by the methods indicated to understand the *kinds* of answers he should seek and the *kinds* he gets when he gets them. Then the teacher may indicate, and stimulate his ingenuity to discover, other ways of getting answers. To illustrate, the recall of the "pairs" to nine and nine suggests answers to additions that are closely related. Thus, "eight and eight are sixteen; so eight and seven are fifteen." Or the pupil may "study a teen," eleven, let us say, thus: He lays out, or imagines, a ten and a one. He moves one from the ten to the one. Now he has "nine and two." Next, he tranfsers one from the nine, and he has "eight and three," and so on. Similarly, he may "study" the rest of the "teens" to eighteen.

To illustrate further, in finding the answer to "nine sevens," let us say, the pupil may learn to think, "I know that six sevens are forty-two, and three sevens are twenty-one; so I add and get sixty-three." Or he may think, "Three sevens are twenty-one; nine sevens are three times twenty-one," and so on.

In any case, the pupil should know the *kind* of answer he seeks, and he should seek and find *his own* answer.

d) Arithmetic requires the pupil to deal with tens and the powers of ten the way he deals with ones, that is, with the single groups to nine. Thus, the pupil counts two tens and three tens together the way he counts two and three together. In each case, he thinks, "Two and three are five." In each case, he writes 5 to show his answer. He writes his ten's answer, using zero to aid, in ten's place. Similarly, he adds hundreds, or thousands, or millions. He adds the way he adds ones, and writes his answer in its proper place. Similarly, he subtracts and multiplies and divides tens and the powers of ten. Similarly, also he multiplies and divides by ten and its powers. In every case, the principle of positional value, through the aid of zero, keeps everything clear. At the outset, while the pupil is learning the principle, the combinations are simple. Later, when the pupil has learned the principle, the combinations are complex. Through a developing mastery of the principle, the pupil makes his work easier and easier as simplicity gives way to complexity.

2	20	200	2000
3	30	300	3000
	0	00	000

(In each case, the addition is the same, "Two and three are five.")

The developing principle extends through the pupil's work with wholes almost from the beginning and may extend also into the study of parts. It may become, as it grows, the organizing center of his work. It conforms his thinking to the characteristic features of the decimal notation. In consequence, he develops the principle and an intelligent use of the system together.

The pupil develops the principle slowly. As his work is commonly arranged, he may add and subtract tens in his second year and take up the multiplying and dividing of tens, including the carrying of tens, in his third year. In his fourth year, he carries his activity to hundreds, thousands, etc., and begins to multiply by tens (later, by hundreds, etc.). Finally, in his fifth year and later, he turns to the process of dividing by tens. His efforts are effective according to the attention he gives to the central principle.

When the teacher supplies his answers, the pupil does not need the principle and, in result, does not get it in his possession. Thereby it means nothing and provides no aid. On the other hand, when the pupil must

supply his own answers, he stands in need of a principle of procedure, and he may quickly note the central thread that runs through the various activities he undertakes.

STUDYING PARTS

e) Arithmetic requires the pupil to study the sizes of parts, to represent them according to their uses, and to combine related parts. The pupil first studies the fractions whose common usage brings them into relation with each other. He starts by considering each alone as a given sized part of a whole: one-half as one of the two equal parts; and so on. From the beginning he must consider size and how to represent size by writing the proper numeral below the line. Here, for the time, the numeral 2, for example, signifies, not two, but half. And from the outset he must think of the fraction as the part, or parts, he has cut out and has before him, and of the expression he writes as "a way to write the *size* and the *number* of the parts."

He deals with the common related divisions: $\frac{1}{2}$; $\frac{1}{4}$ as $\frac{1}{2}$ of $\frac{1}{2}$; $\frac{1}{8}$ as $\frac{1}{2}$ of $\frac{1}{4}$; and so on; $\frac{1}{3}$; $\frac{1}{6}$; $\frac{1}{9}$; $\frac{1}{12}$; and so on. Before he represents any one of a given set, he first makes the proper division and considers it; finally, when he has the idea in mind, he writes. He compares, adds, and subtracts the parts of a given set of divisions—halves, fourths, eights, etc.—making the required subdivisions of the parts of larger size to bring all into parts of the same size before he compares, or adds, or subtracts. Next, he deals with different size parts from different sets—halves and thirds, thirds and fifths, etc.—making the more complicated divisions to reduce to parts of the same size.

He proceeds in multiplying in the order: a fraction by a whole, a whole by a fraction, and a fraction by a fraction. He makes his dividing by a fraction clear by first making his division question clear.

Next, the pupil studies the fractions whose uses do not bring them into such close relations, namely, the decimals. He studies the decimals in terms of their common uses as measures. He measures, first, in *tenths*, counting to nearest *tenth*. Later, he measures, or studies measures, in *hundredths*. Still later, he may merely study measures in *thousandths*. He knows each as a way of measuring with a given accuracy and with a given purpose, and he keeps separate the different measures by distinguishing the different purposes. Thus, he combines—adds and subtracts—measures in *tenths* or measures in *hundredths*, not the two together. Thus, he does not call 10 *hundredths* 1 *tenth*, because the *hundredth* indicates an accuracy of measuring that the *tenth* does not indicate. All this, with his knowledge of the decimal notation, keeps his thinking clear and his calculating in order. His multiplying and dividing of decimals by whole numbers is in line with his earlier thinking and practice.

Keeping in mind the decimals of different sizes as differently used

measures, the pupil can proceed to the multiplying and dividing of decimals by decimals. Thus, if the teacher will set his questions correctly, he proceeds to multiply or divide to find the answer in nearest *tenth,* or *hundredth,* or *thousandth,* as the case may be. Absorbing and following the "rule" of either operation then presents no problem. In his dividing, he uses the caret (∧) to indicate his divisor as a whole number. He thus prepares in advance to find the *kind* of answer he knows he should seek. He thus can carry forward his dividing to the point where his answer is expressed in the desired nearest sized part.

f) Arithmetic requires the pupil to carry his study of the fraction to the point where he can understand and use it as a way to express the relations between quantities. As he learns his fractions at the outset, the pupil may make a good deal of practical use of them. He may deal with the size and number of parts that he can observe and recognize in the simple affairs of the everyday.

Soon, however, the pupil reaches the limit of obvious parts to observe and recognize. There are not many of his everyday affairs that have ideas of parts within them. Here, his study may stop; what is better, it may continue far beyond the observation of the obvious in everyday affairs. It may continue to the point where the idea of the fraction becomes a relational idea. Then a wholly new avenue of studying parts opens for the pupil to enter. The new studies are in terms of what we have come to call "the three kinds of problems": finding the part of a number; finding the part one number is of another; finding a number when a part of it is known. The pupil studies the three kinds of problems, learning to distinguish each from the others, first, when he expresses the fractions as common fractions and, next, when he deals with them in the decimal form. Finally, the pupil deals with the three kinds of problems when he expresses the relational part in *hundredths* only, that is, as per cents.

When the pupil must find the part in the decimal form in the "second kind" of problem, he must know in advance whether he should find his answer in nearest *tenth,* or *hundredth,* or *thousandth.* Next, he may concentrate upon finding the part in nearest *hundredth,* and upon finding each of the two numbers in the relationship when the part in each case is stated as *hundredths.* From this point, dealing with per cents requires little more than learning a different name and sign for *hundredths.*

STUDYING RELATIONS

g) Arithmetic requires the pupil to use his ideas of relational amounts as means to familiarize himself with the nature, import, and consequences of the various social, business, and civic situations and transactions that exist and go on around him.

In earlier discussions we have indicated the ways the pupil may deal with relational amounts in the study of situations.

STUDYING MEASURES

h) *Arithmetic requires the pupil to determine amounts in terms of the standards by which, and in respect to which, we measure them.* "How far?" "How much?" "What weight?" With such questions the pupil must deal. He should learn to deal with them intelligently and confidently. To answer any such question, he takes a unit of measure (a *foot*, a *bushel*, or a *pound*) as a basis, and then he counts the unit *feet*, or *bushels*, or *pounds*. Counting the units is no problem. Learning to be sure what he should count presents the difficulty. In determining amount, every act of effective thinking stems from and reverts to the idea of the countable standard unit. The pupil should learn the standard units. He should learn what a standard is, what the word "standard" means.

The pupil begins by considering the standard units of common everyday usage whose objective representations are known. He learns about them in part by enacting the uses to which they are commonly put. He learns about them in larger part by considering what they are designed to provide, namely, answers to questions about various types of amounts. In consequence, the pupil must make sure of the questions to determine how the use of the standards provides answers. Frequently, the problems of measuring under the simpler conditions of past living stand out more clearly than do the same problems under the complex conditions of the present. The pupil may learn a good deal about present-day standards through their history, through the devious and apparently unusual ways early peoples established and used their crude units of measuring. Though the obvious activity in measuring is counting the units, the hidden and basic activity is consideration of the standard unit.

The pupil should move from the common measures whose standard units have objective representation, the *foot*, the *gallon*, the *pound*, to the less common measures having standard units which he must carry in his mind, as it were. Square measure will illustrate. In this, the standards are unit squares which we do not usually see and handle. For his training in getting in mind the standards, the pupil at the outset must construct and use an objective unit, the square foot, let us say. The usage is roundabout and soon abandoned for short-cut procedures, yet the usage fixes attention on the kind of answers the questions about surface measuring seek. Here again, reference to the history of surface measuring may help to make clear by contrast the present standards.

All such activities as we have been indicating and other similar ones help to make clear the basic, and hidden, question which the pupil must answer in measuring a surface, or in computing its measure. The question is always about the standard unit. In every such measuring situation, the

question is, "Does the surface show squares?" If it does not, the question becomes the problem, "How shall we make the surface show squares?"

Finally, the pupil may become so conscious of the standard unit in general as to be able to deal intelligently with particular standards which cannot be directly applied, as in the case of the cubic units in measuring nonrectilinear containers.

Throughout his study of various standard units, the pupil may learn about each by considering its relations to others. He may be impressed with present relations by studying about how measures have developed from diversity toward uniformity.

SEQUENCES AND PARALLELS

To depict the pattern of the pupil's thinking, we have used the figure of steps, or stages, in seriatim (see p. 31). At that point in our discussion of the nature of number-thinking, it seemed important to emphasize the significance of the fact that the pupil's success in becoming adept at finding his own answers to number questions involving unfamiliar situations follows sequentially upon his achieving the like objective in the first and second steps of such number-thinking. Accordingly, we designated the completed cycle of thinking about and learning the way to deal with unfamiliar number situations as "the single road of number-thinking." We indicated this road, which gives different appearances due to its differing gradients, as a continuous on-going road.

We also noted that progress from the first step through the second and then the third involves an increasing complexity of number-thinking, due to the necessity of recognizing the existence of known, or supposedly known, number relationships among the interesting features and distractions which the so-called practical situations of everyday problems present. In the earlier section of the chapter (p. 31), the relatively simple type of number-thinking characteristic of the first step is illustrated by the diagram representing an experimental method of dividing a group of twelve objects into thirds.

Throughout our discussions we have suggested similar modes of thinking about the enlarging complexities of the number relations the pupil has to learn. For example, the developing idea of dealing with tens the same as with ones helps to bring into clear relations otherwise different relationships. Yet, for us to aid the pupil to gain the use of similar modes of thinking, we need to recognize the existence of the different though parallel relations he has to learn. For the purpose of illustrating the involvement of parallel relationships as these are encountered by the pupil in finding his answers to problems, we suggest the figure of threads, each interweaving with all others and each continuing to run through the resulting patterns.

The threads of the pupil's activity in the course of the normal number experiences of his elementary-school career involve both sequences and parallels with respect to the thinking and learning that enable him to find the answers to the different types of number questions which

THREADS OF THE PUPIL'S ACTIVITY

Activity	School Grade							
	I	II	III	IV	V	VI	VII	VIII
Counting groups.								
Comparing groups.								
Separating and combining groups.								
Combining in the teens.								
Combining into tens.								
Adding and subtracting tens.								
Multiplying and dividing tens								
Carrying tens.								
Adding, subtracting, multiplying, and dividing hundreds, etc.								
Multiplying by tens.								
Dividing by tens.								
Fractions.								
Decimals.								
Hundredths.								
Measures.								
Familiar situations.								
Studying situations.								

Introduction of incidental study
Systematic study and practice

emerge at different levels of school work. These relationships are shown in the accompanying chart, which indicates approximate points of origin of the various threads of activity the pupil must take up and the approximate lengths to which he must carry them for training purposes.

CHAPTER IV

ARITHMETIC FOR PRESCHOOL AND PRIMARY-GRADE CHILDREN

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Introduction

Tommy lined up his toy cars in a row in front of the Christmas tree. Delighted with his new-found wealth in cars, he talked aloud as he gently touched the cars in turn:

Touching these cars:	*He said:*
The first car.	"There's one."
The first two cars.	"One and another one. That's two."
The first three cars.	"One and one and one. That's three."
The first four cars.	"One and one and one and one. That's four."
The first five cars.	"One and one and one, and one and one. That's five."
The first six cars.	"One and one and one, and one and one and one. That's six."

Meanwhile, his twin sister Nancy bounced her new and gaily colored rubber ball in time to this old rhyme:

"One, two; button my shoe.
Three, four; shut the door.
Five, six; pick up sticks.
Seven, eight; lay them straight.
Nine, ten; a big, fat hen.
Eleven, twelve, dig and delve.
Thirteen, fourteen; maids are courting;
Fifteen, sixteen; maids in the kitchen;
Seventeen, eighteen; maids are waiting.
Nineteen, twenty; my plate is empty."

It took Nancy eight trials before she could get all the way through the rhyme. She knew the rhyme, but bouncing the ball forty times in proper rhythm demanded better motor co-ordination and skill than she could muster on the earlier trials.

Superficially, an observer might have thought that Nancy's performance showed more mature knowledge of numbers than Tommy's. She counted all the way through twenty while Tommy used only the numbers

through six. On the other hand, Tommy seemed to be talking in terms of what he could demonstrate with his movements and the cars. Nancy may have understood very well the numbers she used; or she may have used the number names as words in a rhyme with as little relatedness among them as button shoes and a big, fat hen have to each other.

Child Growth and Development in Arithmetic

What children *say*, using number words, is only one of many clues which the interested parent or teacher should use in making observations as to the young child's number maturity. As with older children, glib verbal use of number terms should not be mistaken for certain knowledge of number concepts which a mature person associates with those words. In order to make reliable judgments concerning the maturity of Tommy's and Nancy's number knowledge, one would need to know a good deal about child growth and development in general as well as knowing Tommy and Nancy in particular.[1]

Of course, there is no particular point or satisfaction (except, perhaps, for the proud parent of a superior or supposedly superior child) in making judgments as to the maturity of a child's number concepts unless those judgments help us in our efforts to lead the child on to progressively more mature concepts and understandings.

Arithmetic Readiness

The term "readiness" has been used so widely in the past two decades that it has suffered a common fate of popular expressions in education, that of careless overuse. It is easy to say that a child should be taught reading or arithmetic or spelling when he is ready, but to know when he is ready is in no sense an easy matter.

What Readiness Is. Only to the uninitiated can readiness appear to be a simple matter of reaching some mythical, magical point preceding which the learner is clearly not ready and following which he is clearly and unequivocally ready to learn. Only by the psychologically naïve is readiness conceived as being a boundary line across which the learner

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[1] See, for example:

Arnold L. Gesell *et al., The First Five Years of Life.* New York: Harper & Bros., 1940.

Arnold L. Gesell and Frances L. Ilg, *The Child from Five to Ten.* New York: Harper & Bros., 1946.

Arthur T. Jersild and Associates, *Child Development and the Curriculum.* New York: Bureau of Publications, Teachers College, Columbia University, 1946.

Staff of the Division of Child Development and Teacher Personnel, Commission on Teacher Education, American Council on Education, *Helping Teachers Understand Children.* Washington: American Council on Education, 1945.

steps at a clearly defined time from "Unreadiness Land" to "Readiness Land." The attainment of readiness is a continuing process of becoming more ready than one was previously. Adults will understand the learning of children much better if they will think in terms of their being more ready or less ready rather than ready or unready.

Further clarification of the readiness idea is dependent upon a realization that there is little point in extended talking or thinking about readiness in general. Learning readiness is always readiness for learning particular content or learning in a particular area. We need to recognize that a child may be quite ready to learn so far as his reading activities are concerned while being less well prepared for learning in arithmetic. Further, within the area of arithmetic, the child who is quite ready for learning simple addition combinations may be lacking in background for learning how to add two three-place numbers when carrying is involved.

With young children this question of "readiness for what?" needs particular attention for the simple reason that adults who guide the learning of children are inclined to assume that children know "what" they are working into even when they have little or no notion about it. Primary-grade children are sometimes hindered in their learning as they are introduced to arithmetic as a school subject because they think of it as some strange and new learning task instead of as a simple extension of experiences they have already had with number and quantity.

Finally and obviously, readiness is an individual matter. For a parent to expect four-year-old Susie to count to 100 by rote "because her older sister Jane could do that when she was four" is as unfair and unwise as it is for a third-grade teacher to expect all her pupils to be ready at the same point in the school year for telling time to an exactness of five-minute intervals.

Assessing Children's Readiness for Number Experiences and Instruction. Children supply parents and teachers with an abundance of clues concerning readiness for number ideas and concepts if the parents and teachers understand children well enough and are observant enough to note the clues.

Take, for example, three-year-old Richard. His aunt gave him a nickel. Then his uncle offered to trade, giving him a dime for a nickel. Richard refused, saying, "Mine's bigger." The uncle explained, "Yes, your nickel is bigger; but this dime will buy more. If you go to the store for candy, you can pay the man this dime. He will give you a nickel's worth of candy and a nickel, too. This dime is worth *two* nickels."

Richard pondered this mystery and decided to make the trade. Pragmatically, he ambled off to the corner grocery, where he told the clerk, "This is a dime. It's worth two nickels. You're supposed to give me a

nickel's worth of candy and a nickel, too, for this dime." Richard got his candy and his nickel. But that was not all; Richard had shown very clearly by his behavior that, although his concepts of the comparative values of coins were relatively immature, he was clearly ready to profit from the instruction and the ensuing experience which made his concepts of "nickel" and "dime" more mature.

Parents and teachers in general need to spend more time talking with children. Given a chance to ask questions and to express the ideas they have and being encouraged by questions and comments to elaborate on their thoughts about number, children will reveal much about their present number concepts. For example, when a first-grade child says that he ate a million cookies, it may be time well spent for an adult to inquire further into what the child really means by a "million cookies."

For more exact surveying of children's readiness for beginning systematic instruction in arithmetic, various helpful guides are available.[2] Careful study of such materials will add much to the teacher's background for judging children's readiness for various phases of instruction in arithmetic.

Developing Readiness. Authorities on the teaching of arithmetic are rather well agreed that the teacher's responsibility with regard to readiness is just begun with an appraisal of the readiness of individuals and class groups for instruction in any particular phase of arithmetic, e.g., the subtraction combinations or multiplication of a two-place number by a one-place number with carrying. Appraisal should be a forerunner and a concomitant of the highly important business of developing readiness where it is lacking. If a child or a group of children do not seem to possess adequate readiness, the teacher's next step is one of positive action rather than passive waiting.

Let us take the case of readiness for learning the simple addition and subtraction facts, e.g., those with sums and minuends of ten or less. If primary-grade children are to stand a good chance of understanding the various combinations in this set of addition and subtraction facts (that is, if they are to be considered ready for this learning experience), they should first have acquired certain skills and understandings concerning both ordinal and cardinal numbers from one through ten. They should

[2] William A. Brownell, *Arithmetic in Grades I and II.* Duke University Research Studies in Education, No. 6. Durham, North Carolina: Duke University Press, 1941.

Leo J. Brueckner and Foster E. Grossnickle, *How To Make Number Meaningful*, pp. 56–58. Philadelphia: John C. Winston Co., 1947.

Anita Riess, *Number Readiness in Research*, pp. 8–10. Chicago: Scott, Foresman & Co., 1947.

Lucy L. Rosenquist, *Young Children Learn To Use Arithmetic*, pp. 161–69. Boston. Ginn & Co., 1949.

be able to count to ten (preferably, further) by rote and when enumerating objects; they should be able to read and write the numbers one to ten; they should know the "next larger" and "next smaller" numbers in relation to any of these numbers as well as knowing which of these numbers are smaller or larger than any other given number in the series. They should have clear ideas of groups of objects comprising ten or fewer items; they should be able to give the number names corresponding to various sized groups of concrete objects and semiconcrete representations; they should be able to reproduce such groups of ten or less when given the number name; they should be able to compare such groups accurately in terms of more or less, larger and smaller. Furthermore, they should have enough facility in expressing number ideas so that they can tell what is being done when groups of objects are combined to make a new and more inclusive group or when a more inclusive group is broken up or separated into new smaller groups.

What if children do not have these stated prerequisites for reasonable success in really understanding the addition and subtraction process as exemplified in the specified set of number combinations? The answer is, obviously, to set about building them.[3]

Prevention of Unnecessary Frustrations

Arithmetic is a part of the life of the normal person (child or adult). Achieving competence in arithmetical thinking, understandings, and skills can be a type of insurance against the frustrations which come to those who are incompetent in handling situations involving number. Too many older children and adults now suffer from feelings of inadequacy, inferiority, and other frustrations because of their early unfortunate experiences with arithmetic. Too often, those early experiences have been unfortunate and positively detrimental as a direct result of ineffective and confusing instructional procedures to which the children were subjected.

Traveling from the Known to the Unknown. What may seem like a fast start in arithmetic to many parents and teachers of young children may really be a slow start in terms of later progress in handling arithmetical ideas. Children need to become familiar with preliminary concepts before they can be expected to deal successfully with later concepts which rest upon earlier ones. If each new idea can be presented so that the learner sees it as a clear and orderly extension of something he already knows and understands, he is saved the bewilderment and even panic which is so natural when one new and strange concept after another is forced upon him.

[3] Instructional suggestions are given in a later section of the chapter.

Counting in progressively more mature ways should run right along with instruction in the four fundamental processes, providing for children one of the familiar and reassuring "knowns" which will make the new and previously "unknown" seem somewhat familiar and comfortable too.

Maintaining a Challenge. The caution not to go too fast too soon may be carried to an extreme if one is not careful to remember that children also need to be challenged. If parents and teachers could get away from the fixed notion of "one right way" to solve every arithmetic problem, they would be on the way toward opening up new avenues of arithmetical thinking for children. If children see arithmetic as a routine series of manipulations applied by hit or miss techniques until somehow the "right" answer emerges, it is inevitable that many of them will be frustrated and discouraged by the whole thing.

Only at first glance does it seem like a contradiction to say that presenting arithmetic as a challenge to the child's best thinking makes it easier for him. Most children like to "figure things out" for themselves, provided they have enough ability, skill, and understanding to be successful in their efforts at seeking solutions. They enjoy unlocking a problem situation, provided they have the requisite keys to that situation. On the other hand, if there is only one right way—the teacher's way or some other adult's way—the child becomes so involved in finding that way and avoiding the penalty for finding a different reasonable solution that the joy of working out an original solution in which he can feel the pride of personal accomplishment is lost. And is it not a truly tragic and unnecessary frustration for a child to work out a problem his way, a correct way, only to be reprimanded and have the solution counted "wrong" because it does not conform to the particular pattern used by the teacher or the textbook?

Content and Teaching Method in the Early Years

The way in which a child learns arithmetic affects not only his competence in arithmetic but also his general competence in living. The personality by-products of subject-matter learning are tremendously important even though not always clearly or immediately apparent. The acquisition of a well-organized understanding of arithmetic and its various interrelationships should make a positive contribution toward better personality development for the person who possesses such equipment for solving the number problems he meets in his everyday affairs.

If learning in arithmetic is to eventuate in desirable results so far as both personality and arithmetic knowledge are concerned, two important questions need to be answered: (*a*) *What* should be taught in the preschool and primary-grade years? (*b*) *How* should this material be taught?

Age and Grade Placement versus Sequence of Experiences and Ideas

Conscientious parents of preschool children often inquire: What should we teach our child about numbers before he starts school? How far should he be able to count when he is five years old? Should we teach him easy addition facts? Just what should he know if he is to get along all right when he is in the first grade?

When their children are in the first grade, the same parents are apt to ask the child: Have you had any arithmetic yet? Do you practice writing numbers? How far can you count now? Did you have any arithmetic today? Are you learning to add (or subtract)?

Or parents of second- or third-grade children may say: I can't understand the schools now-a-days. Jack says he didn't have any written arithmetic work today or yesterday. I don't want him to be behind in arithmetic. When I was in school, we had had lots of addition (or subtraction, or multiplication, or division) by the time we were in his grade.

Meanwhile the teachers are also asking questions of themselves and of others: How much arithmetic should be done in the grade I teach? How much should we cover? Should second-graders learn all the addition and subtraction combinations before they are promoted to the third grade? The third-grade teacher seems to expect it, but some of my pupils can't learn them very well.

So run the questions, on and on—serious questions by sincere questioners. They want their questions answered, and answered definitely. Most of the questions indicate that those who ask expect a simple, direct answer to questions which have no simple answer applicable in all situations or for all children. How far should a child be able to count when he is five years old? The answer depends on the child, his past experiences with number, his mental maturity, and other matters which vary among individual children. The answer depends also upon what is meant by the word "count." Or take this question: Should second-graders learn all the addition and subtraction combinations before they are promoted to the third grade? The answer depends upon who those second-graders are; variations among second-graders in general are much too wide to allow a simple "yes" or "no" answer for all. The question makes a person wonder if the questioner should not ask whether all second-graders *can,* rather than *should,* learn what is implied. And again, what is meant by "learn"? Does it mean permanent learning or temporary accomplishments? Does it involve understanding the combinations and the processes or just saying answers?

To say that these questions about what arithmetic children should learn in the preschool and primary-grade years cannot be answered by

categorical, neatly packaged prescriptions is not enough. Parents and teachers need help as to what they should expect of children and what they should seek to teach children. That help will not come from adherence to rigid age or grade standards of accomplishment. It will come, rather, from careful consideration of: (*a*) research on typical performance and range of performance by large groups of children of different ages; (*b*) study of the readiness of individual children for particular subject matter and experience; (*c*) understanding some of the logical sequences peculiar to the subject of arithmetic; and (*d*) considering the effect of teaching method upon the ease or difficulty of learning. The first two of these points have been briefly discussed and references pertaining to them have been cited earlier in this chapter. The balance of the chapter will deal with the latter two points.

What a child or a group of children should be taught at any particular time is not so much a matter of teaching a certain piece of subject matter when the children are a certain age or in a certain grade. It is much more a matter of planning subject matter and teaching method so that each learning experience follows naturally upon preceding learning experiences and results. Arithmetic is a systematic area of knowledge with very clear lines of sequential development. The logic of the subject is so much a part of arithmetic that effective arithmetic teaching must take into account how certain ideas and concepts grow out of and are built upon certain other ideas and concepts. Limitation of space makes it impossible to consider all the content of preschool and primary-grade arithmetic learning, but enough examples can be given to demonstrate that parents and teachers can help children learn arithmetic much better when they (adults) carefully consider systematic and sequential order of arithmetical ideas and when they present those ideas in ways which aid children to understand the content of their arithmetic activities.

Meanings of Numbers

While in practice it is impossible to draw sharp lines of distinction between development of the meanings of numbers and development of the number processes, the two are separated here to facilitate emphasis upon some of the crucial matters of arithmetic content in these two areas. At least six phases of the meaning of numbers are crucial considerations in the thinking of those who attempt to guide young children in their learning of arithmetic. We do an injustice to the child if we slight any of these: rote counting, ordinal number, cardinal number, the reading and writing of numbers, place value, and zero.

Rote Counting. Learning to say the number names in order, i.e., rote counting, is a significant early experience which every child needs to prac-

tice a good deal. The father who says of his five-year-old son's rote counting, "He's *only* counting *by rote*. He doesn't know what the numbers *mean*," is perhaps making as much of a mistake as the one who says, "He can count as fast as anything all the way up to 100. He's going to be a whiz in arithmetic." Ability to count by rote is by no means an indication of comprehensive understanding of numbers; but it *is* a necessary beginning. Children do need to know the names of the numbers with which they will deal in arithmetical situations. Furthermore, they need to know those names in order.

Ordinal Number. The importance of ordinal number is not understood by many parents and teachers. In fact, they often try to lead the children from rote counting to cardinal number (answering the question: how many in all?) with little or no intervening attention to ordinal number (answering the question: which one?). Of course, some practice in ordinal counting does necessarily take place because the children cannot answer the question "how many?" for most number groups without using the process of ordinal counting. Few children can recognize groups larger than four without counting the objects in the group serially first.

When teachers do pay attention to the ordinal idea (position in a series), they usually use the terminology *first, second, third,* and so forth. This is correct terminology, but the use of the number names *one, two, three,* and so on is equally correct and more commonly used in expressing the ordinal idea. Susan gets up at *7* o'clock in the morning and leaves home for school at *8* o'clock. She walks down *Third* Avenue to go to school, where she is in the *third* grade. Her class is called Section *2* of Grade *3*. Their classroom is Room *4* (the *fourth* room as one goes down the hall from the front entrance). Susan removes her coat, hangs it in Locker *7* and goes to her desk, which is the *second* desk in the *fifth* row. No other children are in the room, so she knows she is the *first* child to arrive this morning. She takes out a book so she can finish the story she was reading yesterday. She remembers that she was on page *15* when she closed the book, so she opens it to that page.

Both terminologies are used in this account of Susan's activities, but every number name in the account expresses an ordinal rather than a cardinal idea. Each answers the question "Which one?" rather than "How many?" Which hour? Which avenue? Which grade? Which section? Which room? Which locker? Which desk? Which row? Which child? Which page?

Cardinal Number. The cardinal number idea will be better understood against the background of clear understanding of ordinal number. When children want to know how many, they count objects in order (ordinal

idea) and then announce the last number name they use as a cardinal number.

Even when a child is taught to compare two groups as to size (e.g., 24 in one group and 32 in the other), he may be dealing with cardinal numbers which tell how many in each group, but his understanding of the comparative size of the two groups is certainly increased by his knowledge of the relative position of 24 and 32 in serial order. If he is writng out his subtraction, he writes 32 first. Why? Because he knows that 32 comes later in the number series than 24, i.e., that it is bigger.

Children will actually get clearer ideas of cardinal number if they first understand ordinal number and then carry on and expand both ideas together. The terminology "ordinal" and "cardinal" need not be mentioned to young children. They can be taught the distinction very nicely in connection with the simpler "Which one?" and "How many?" vocabulary.

Reading and Writing Number Symbols. The too-prevalent early stress upon reading and writing number symbols perhaps grows out of two common weaknesses in instructional programs. One is that adults of the present generation are too prone to think of arithmetical computations as necessarily written. The other is that when teachers or parents lack a broad basic knowledge of fundamental number concepts needed by children and how to build them, they seize upon reading and writing numbers as "something to be doing" in arithmetic. This criticism is not intended to suggest that reading and writing numbers be postponed until children's number ideas are relatively mature, but rather that children should not be instructed in the reading and writing of numbers in isolation from the development of number meanings. Natural occasions for reading and writing numbers which occur early in a child's school life should be utilized. For example, pupils need to write numbers to record numerical features of everyday situations, such as how many children will stay at school for lunch, or what date it is.

Notation and Place Value. Place value is regarded by many primary teachers as being outside their province. They think of it as becoming important only when children deal with large numbers or with "decimals," meaning decimal fractions and decimal mixed numbers. It never occurs to them that as soon as we record numbers of 10 or larger we are dealing with a notational system which happens to be a decimal system. Nor do they realize the importance of helping children understand this decimal system (at their level, of course) from their first uses of Hindu-Arabic notation with a base of ten. For example, the repetition of nine basic number names (one to nine) in rote counting and in the reading and writing of numbers above ten provides early contact with the system, and children can profit from a discussion of the fact that they "start over again"

every time they pass twenty or thirty or other even-decade numbers in counting.

The numbers in the "teens" decade become especially important when children learn addition and subtraction combinations with sums and minuends of ten or more. Time spent on the meanings of the number symbols in one's place contrasted with the ten's place is definitely well spent. Devices like the abacus are very helpful at this point.

Or let us go on to a later phase of addition, carrying. Children can and do learn the mechanics of carrying in addition, just as they can and do learn to give the answers to the addition facts with the sums of ten or more, even though they do not see the meanings of what they are doing. But they cannot *understand* carrying in addition, or sums of more than 10, without the concept of place value. The common classroom terminology says that "we carry 1" with no suggestion or demonstration that it is 1 *ten* that is being carried and that we have 1 ten to carry because we first change 10 ones into 1 ten. Perhaps it is not too late to build up this relationship between the place value of ones and that of tens when carrying is introduced; but certainly children will be far better prepared to see the significance of the carrying process if the idea of place value has been developing all along.

There can be little doubt that many pupils have a hard time in later work with decimal numbers because they were not taught the place-value concepts which could so easily have been introduced in connection with simpler numbers and operations.

Zero. Of course, our notational system would not work out properly without a place-holder to be used to indicate no ones or no tens or no hundreds in a number which has quantity to be expressed in other places to the left of the one with no quantity indicated. Extended theoretical discussion of zero as a place-holder is not needed in the primary grades, but zero's meaning, "no ones," comes in as soon as we discuss the meanings of two-place numbers like 10 and 20.

The development of the zero concept provides an excellent illustration of how matters of sequence apply to all six phases of number meanings discussed above. The order of these six topics is a general order only. It applies most clearly as an order of *introduction* of ideas. Zero comes last on the list because it should be introduced last. The symbol and the word for "zero" with its meaning, "no ones," has no role to fill until the meaning of place value in even-decade written numbers is developed. The so-called "zero combinations" in addition and subtraction are usually not needed until they are introduced in connection with the addition and subtraction of two-place numbers.

The point has been made earlier that it is impossible to make an exact

age or grade placement for arithmetic content which will be appropriate for all children. In general, however, rote counting, ordinal counting, and cardinal counting will be started by practically all children before they enter the first grade; but their competence and understanding will vary widely. Some children can do some reading and writing of numbers when they come to the first grade; many will begin those experiences in the first grade. All four of these phases of number meanings will continue to be developed all through the primary grades as the teachers guide learners to progressively more mature concepts. The meaning of place value and zero can reasonably be taught to some pupils earlier, but for most children perhaps these will be developed largely in the third grade. These rough approximations of placement must be modified to fit the varying abilities and backgrounds of individual children. They must also be varied in the light of changing teaching methods. *How* certain content is taught has much to do with *when* it is appropriately taught.

Aside from the sequence of introduction of ideas, there is much overlapping in the development of these six phases of number meaning. One topic is not completed before another is begun. Rote counting is not completed before ordinal and cardinal counting are begun. A child may first count to 10 by rote, next count 10 or fewer objects in order to designate which one or how many in all, and later extend his rote, ordinal, and cardinal counting to larger numbers. Each of the earlier concepts is continuously expanded and extended alongside of and in relation to the ideas which are introduced later. Often several ideas are so interwoven as to be inseparable if a number experience is to be satisfactorily understood by the learner. The same type of interrelationship applies to number processes, among themselves and in relation to number meanings.

The Fundamental Number Processes

To many people, arithmetic is synonymous with the four fundamental processes of addition, subtraction, multiplication, and division. They represent what we do with number and how we operate within the number system. They cannot, therefore, be separated from number meanings except for purposes of discussion.

Meanings of the Fundamental Processes. The most significant modern emphasis in regard to the four fundamental processes is that all four are basically processes of regrouping. This generalization, once it is understood, adds meaning to each process at the same time that it simplifies each and the interrelationships of all.

An earlier section of this chapter has already characterized some relationships of all the processes as extensions of counting. In fact, any problem which can be solved by adding, subtracting, multiplying, or dividing

can also be solved by counting. The larger the numbers which are involved, the more cumbersome does counting become and the greater is the simplification which results from use of the four fundamental processes of arithmetic. In the early grades, while the numbers being used are still comparatively small, there are very real advantages in having children solve problems occasionally both by counting and by the appropriate fundamental process. The aims of this practice are (*a*) that the children may see the relationship between the fundamental processes and counting; and (*b*) that the comparative efficiency of the fundamental processes may become apparent.

If the primary-grade teacher views addition, subtraction, multiplication, and division as processes of grouping and regrouping, she will also see the importance of providing much grouping experience for the pupils. There would be less haste in presenting the number combinations in their abstract forms if parents and teachers realized that experience in grouping and regrouping concrete objects and semiconcrete representations (e.g., dot patterns) are real arithmetical learning situations for children, that time so spent is not time wasted, and that understanding acquired from these experiences in manipulating groups is not a "frill" but a prerequisite for later easy understanding of necessary abstractions.[4]

The proper meanings of the fundamental processes as they should be introduced to children can perhaps be emphasized by calling attention to common misconceptions which children often acquire either because of inadequate teaching or because of actually wrong teaching.

Addition is *not* a process of getting more. It is a process of *putting together*. The sum in addition is not more than the combined addends. When 4 chairs in one group are moved over and combined with a group of 5 chairs, we can say we *added* the 4 chairs to the original group of 5; but there are not more chairs. Two smaller groups have been combined into a new group which is larger than either of the original groups, but there were 9 chairs all the time.

Subtraction is *not* a process of getting less. It is a process of *taking apart* or *separating*. When Joe has 8 color crayons and gives 3 of them to Sammie, Joe will then have fewer color crayons, but there are not fewer crayons altogether. The fundamental process is that of separating Joe's original group of 8 color crayons into two new groups, a group of 3 which he gives to Sammie and a group of 5 which he keeps.

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[4] A common misinterpretation suggests that "concrete" is synonymous with "meaningful" or "easy" and that "abstract" is synonymous with "meaningless," "vague," or "difficult." No such use is intended here. The essential abstractions of arithmetic are generalized terms, ideas, and relationships which are derived from varied particular experiences and can be considered and used apart from the material or concrete objects associated with the particular situation.

Similarly, multiplication is *not* a process of getting more and division is *not* a process of getting less. Multiplication is a process of adding (putting together) two or more groups of the same size to form one larger group; the number of items involved before and after multiplication is the same. Division is a process of separating (taking apart) a larger single group into two or more equal-sized groups; the number of items involved before and after division is the same.

Interrelationships among the Four Fundamental Processes. If the processes are developed as simple matters of grouping and regrouping and if children are given ample opportunity to do the grouping and regrouping, there is no reason why the typical child should leave the third grade without seeing these relationships: (*a*) Addition and subtraction are opposite processes—putting together and taking apart. (*b*) Multiplication and division are opposite processes—putting together and taking apart. (*c*) Multiplication is usually a faster putting-together process than addition. (*d*) Multiplication always deals with equal-sized groups, while addition may deal with either equal-sized or unequal-sized subgroups. (*e*) Division is usually a faster taking-apart process than subtraction. (*f*) Division always deals with equal-sized groups, while subtraction may deal with either equal- or unequal-sized subgroups.

The Fundamental Processes in Computation and in Problem-solving. In textbooks, workbooks, and standardized tests, and in discussions of arithmetic activities, common usage of the terms "computation" and "problem-solving" seems to indicate a sharper contrast or distinction between them than the views expressed above would necessitate. As the terms have usually been applied, the distinction seems to have been this: In computation the number process to be used is indicated, while in problem-solving the "solver" must decide what process is appropriate before performing the computation. If the four fundamental processes are really understood—separately and in relation to one another—*how* to add, subtract, multiply, and divide (i.e., computation) should certainly involve pretty much the same understandings as are needed in knowing *when* to add, subtract, multiply, or divide. When a child can subtract, in the computational sense, but does not know when subtraction will solve or simplify a number situation, does he really know the subtraction process? Is it even correct to say that he can subtract?

Teaching the Meanings of Numbers and Processes

Whether or not such arithmetic content as that suggested above is appropriate for primary-grade children depends upon their readiness and the effectiveness of the teaching. It is only fair that children be given the best instruction we know how to provide. Among the recommendations

for carrying on an effective instructional program for primary-grade children, certainly these should be included: (*a*) The teacher should herself possess a good understanding of arithmetic. (*b*) Parents should be informed as to practical ways in which they can help. (*c*) The numerical aspects of everyday experiences in and out of school should be utilized and expanded. (*d*) The classroom should be equipped so as to aid learning. (*e*) The learners should play an active rather than a passive role.

The Teacher's Understanding of Arithmetic. The teacher should herself possess a good understanding of arithmetic and of teaching aids and should study the suggestions which are increasingly available in the professional literature. If young children are to get the best foundation for more advanced arithmetical content in the higher grades, primary-grade teachers must know arithmetic in general, not just that content usually presented in the earlier grades. How can a teacher lay a good foundation for later work with common and decimal fractions and percentage if she does not understand that material and consider what she can be doing now to build readiness for later concepts?[5]

Not only does the primary-grade teacher need to study the subject of arithmetic in order to lay proper foundations for children's later learning but she also needs, very often, to understand better some of the primary-grade content which, through no fault of her own, she never learned adequately herself.

Take, for example, the three types of situations in which subtraction is appropriately used to arrive at a solution: (*a*) the decomposition, or "take away" situation; (*b*) the comparative, or "how many more" situation; and (*c*) the additive, or "how many needed" situation. Large numbers of primary teachers present the "take away" situation as being a comprehensive coverage of the meaning of subtraction because they have not noticed the significant differences between it and the other two types. Simultaneously, they wonder why children have trouble knowing when to subtract in problems.

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[5] Available professional books provide a wealth of comment on basic principles and specific suggestions for teaching:

B. R. Buckingham, *Elementary Arithmetic: Its Meaning and Practice.* Boston: Ginn & Co., 1947.

Brueckner and Grossnickle, *op. cit.*

R. L. Morton, *Teaching Arithmetic in the Elementary School,* Vol. I, *Primary Grades.* New York: Silver Burdett Co., 1937.

Rosenquist, *op. cit.*

Herbert F. Spitzer, *The Teaching of Arithmetic.* Boston: Houghton Mifflin Co., 1948.

Catherine Stern, *Children Discover Arithmetic.* New York: Harper & Bros., 1949.

Harry G. Wheat, *The Psychology and Teaching of Arithmetic.* New York: D. C. Heath & Co., 1937.

Consider the variations of the subtraction idea which would be apparent to both teacher and pupils if they were to act out what happens in each of these subtraction situations:

1. Bobby has 8 marbles. He gives his little brother 5 of his marbles. Now how many marbles does Bobby have? (Or: How many does Bobby have left?)

2. Bobby has 8 marbles. His little brother has 5 marbles. How many more marbles does Bobby have? (Or: How many fewer does his little brother have? What is the difference?)

3. Bobby had 8 marbles. He dropped some on the floor. He still has 5 in his hand. How many should he look for if he is to get them all back? (Or: How many are gone? How many are needed?)

In acting out the first situation, Bobby will begin with 8 marbles or whatever he uses to represent marbles. He will hand his little brother 5 of the objects. He will actually give them away or remove them from the remaining 3 (the remainder).

In acting out the second situation, 13 marbles or other objects will be used. Eight objects will be lined up in one row, 5 in another, so they can be compared. Five objects of the group of 8 will be matched against 5 of the other group, showing 3 left over (the difference).

In acting out the third situation, Bobby will begin with 5 objects. He will add 3 more objects to the group of 5 (by counting or by bringing in a group of 3) to make a total of 8.

Lest the differences in action for these three situations lead too far afield from the basic likeness of all subtraction situations, the teacher should also note and help children to see wherein all of these situations involve taking apart or separating. In the first instance a group of 8 marbles was separated into groups of 5 and 3. In the second case a group of 8 was separated into a group of 5 (which matched up with another group of 5) and a group of 3 (which was left over after the matching). In the third case the original group of 8 was separated into a group of 5 (still present) and a group of 3 (not immediately present). This third situation is most easily presented to children by using empty spaces to represent the items not present, e.g., a 6-space or 12-space egg carton with some spaces filled and some empty.

If the teacher does not understand these three subtraction situations clearly, she cannot properly guide children to see that all three are subtraction situations. That is, it will be no wonder that children do not know they should subtract in situations 2 and 3 if they have been taught that subtraction always means "take away" (situation 1).[6]

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[6] These three ideas should, of course, be introduced one by one, with enough intervening experience with each idea so as to make the similarities and differences clear to the children.

Keeping Parents Informed. Parents of younger children are often eager to be instructed in ways they can help the children with number-learning. They will not intentionally confuse or misdirect children's thinking. They often "do what was done to themselves," which may or may not be helpful.

Parent-teacher conferences will be of real value to both. The parent can help the teacher by supplying the background concerning the child. The teacher can help the parent by indicating wherein the child needs help and how to give that help.

Group meetings with parents are also helpful. The teacher can discuss with the parents what the school is trying to do in its informal and formal arithmetic program, explain the reasonable expectations from children, and offer suggestions as to how to help children expand their number meanings.

Another constructive aid may be supplied to parents by distributing materials which they can take home and use for reference. An alert faculty group can work out such materials co-operatively, and perhaps learn much themselves in the process of deciding what to suggest to parents.[7]

Utilization and Expansion of Everyday Number Experiences. Edwina Deans has summed up a practical position with regard to incidental versus planned number experiences.

> The argument of whether number work at the primary-grade level should be planned or incidental is useless. Thoughtful consideration of numbers and the number system, of the child, of the world in which he lives, and of the ways in which he learns points to the conclusion that both are necessary and that neither alone is adequate.
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> . . .
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> The teacher is faced with the necessity of deciding on the worth of the experiences for the group as a whole or of selecting those experiences which will be of value. If the selection is made wisely, children will not be wasting time on those aspects with which they are familiar. Neither will they be asked to delve into those phases for which a previous background or experience or readiness has not yet been provided.[8]

A question frequently asked by parents and teachers is this: Should children have arithmetic in the first and second grades? Of course, they

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[7] A good example of such materials is provided by Herbert F. Spitzer in *Primary Arithmetic Bulletins for Parents.* These include: "Use of Number in Conversation," "Counting," "Simple Measurement," "Number System," "Easy Addition Facts," "Easy Subtraction Facts," "Hard Addition Facts," and "Hard Subtraction Facts."

[8] Edwina Deans, "The Practical Aspects of Number Work at the Primary-Grade Level," *Arithmetic, 1947,* pp. 17–18. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947.

should have arithmetic. They do have arithmetic every day of their lives. Our responsibility is (*a*) to help them get the optimum benefit from the arithmetical content of their everyday experiences in and out of school, and (*b*) to introduce planned experiences with number ideas and processes.

In the first grade, children's number experiences should be chiefly but not entirely of the informal type. The more carefully preplanned number experiences increase in proportionate emphasis throughout the early grades until they may take up the major portion of the arithmetic process in the third grade. The present writer recommends, further, that third-grade children have a definitely scheduled daily arithmetic period. The main point, however, is not whether experiences are incidental or preplanned or whether they are labeled informal or formal but, rather, whether they make sense to the children and contribute to increasingly more mature understanding of number.

A second-grade group sang "Ten Little Indians." The teacher suggested that they sing it again and act it out. She had one child count out ten little "Indians" to come to the front of the room. Each one knew "his" number (ordinal concept) and that there were ten in all (cardinal concept). The ten little "Indians" had chairs to sit on. As the song was sung, they stood up or sat down to fit the forward and backward counting in the song. After enjoying this experience a few times, the children were asked, "What do we do when we say 1, 2, 3 or 10, 9, 8, and so on?" One child said, "We counted forward and backward." After a little more discussion, another child said, "We added one every time when we counted forward and we subtracted one every time we counted backward." No doubt, the individual children in this group varied considerably in what they learned from this experience and in the maturity of their concepts when it was over, but all were perhaps benefited to some extent and in relation to their readiness to benefit.

Similarly, the skilful teacher will realize the opportunity to add to number meanings in connection with the use of page numbers in books: This is page 15. What page number comes next? One story begins on page 24, another on page 42. Which one comes first? The book has 204 pages; our story begins on page 98. Shall we open our books near the front, in the middle, or near the back?

One first-grade classroom has 6-inch square asphalt tiles on the floor. Varied colored tiles make a design. One morning before school started the alert teacher noticed children stepping on the tiles of certain colors, counting as they stepped on each. That morning the children had what they thought of as a "lot of fun"; the teacher saw to it that they had a lesson in counting and grouping.

In another elementary school, the different class groups from the second grade through the sixth grade take turns in operating a school-supply business for the convenience of all children in the school. Tablets, pencils, and notebook paper are purchased at a reduced price from a local merchant and resold at the school at the regular retail price. In the process of managing the school-supply business, the children get practice in simple number operations; they learn how to make change correctly and efficiently; they learn something about taking inventory and keeping records; and they get an elementary idea of the meaning of profit on sales.

A second-grade group takes its turn to sell school supplies near the end of the school year. They have had experience for several months in going to the "store" in other classrooms to make individual purchases of materials, so they have some general ideas of the total situation from the viewpoint of the customer. They have also had varied informal number experiences in the classroom which have added to their competence in handling numbers. Now they feel very grown-up and important to be running the store and to have older children coming to buy from them.

The teacher and the children do some very careful advance planning. Of course, the teacher of the second-grade group has to take a much more active part in the planning and operation of the supply sales than is necessary with older children; but the children do have a chance to do as much of the planning as possible.

Some children are better prepared than others to serve as clerks. They get their turn first. The less mature pupils (so far as arithmetic is concerned) get their turns later, when they will have a greater chance of having a successful experience. The teacher knows the children well and sees to it that the very weak get the easiest jobs and that the more able pupils are at hand to help when needed. All emerge from the "storekeeping" more mature in number meanings and in personal assurance and adjustment.

Rosenquist describes carefully planned experiences which use a classroom situation to give understanding of and practice with grouping and regrouping as a readiness-building experience which will help children in later learning of addition and subtraction facts.[9]

Sometimes the situation "happens" and the teacher uses it wisely; sometimes the teacher makes it happen through careful advance planning. In either case, number knowledge can be expanded.

Equipping a Classroom for Learning. Concrete aids in the teaching of arithmetic are especially helpful in guiding the learning of young chil-

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[9] Rosenquist, *op. cit.*, pp. 102–3.

dren. The topic has been so well treated elsewhere[10] that extensive treatment at this point is unnecessary. A few suggestions concerning the use of concrete aids in the teaching of place value and the decimal system may suffice.

Money—so interesting to most children and adults alike and so neat a concrete aid in the teaching of the decimal idea and place value—is not used as much nor as well as it could be in most primary-grade classrooms. If objections to the use of real money arise, toy money can be purchased or made at small expense.

A teacher planned a simple series of experiences with first-semester first-graders which provided a setting for developing meaningful reading and writing of the number symbols and also gave the children a start on the decimal idea and place value. One Monday morning when the children came to school, there was a penny fastened to a card on the bulletin board with Scotch tape. Underneath it was a large figure "1." The next day there were two pennies and the symbol had changed to "2." So it went; every day one more penny and always a new symbol, the cardinal number for the group of pennies on display.

By the second Thursday, the group of pennies had grown to 9. All along, the children were curious and interested, delighted to discover how the group grew each night, and soon predicting how many pennies there would be the next day. On the second Friday, there were 10 pennies and the number "10." To the teacher, this was a crucial point. She could have had the children read the number and let it go at that. But she went on, asking if anyone could tell her some other piece of money that was worth the same as 10 pennies. Several children knew that would be a dime, and the teacher was prepared to produce one as soon as it was mentioned. She pointed out that the 1 in the 10 stood for 1 dime instead of 1 penny as on the first day. The germ of the decimal idea and of place value had been planted for at least some of the first-graders.

Dimes and pennies are helpful learning aids in such situations as these: for sums over 10 (e.g., $5 + 8 = 13$ demonstrated as a group of 5 and a group of 8 changed to a group of 10 and a group of 3); for bringing out generalizations related to 10 (e.g., when 9 is added to another number,

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[10] Irene Sauble, "The Enrichment of the Arithmetic Course: Utilizing Supplementary Materials and Devices," *Arithmetic in General Education*, pp. 157–95. Sixteenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1941.

Foster E. Grossnickle, "The Use of Multisensory Aids in Developing Arithmetical Meanings," *Arithmetic, 1948*, pp. 1–14. Supplementary Educational Monographs, Chicago: University of Chicago Press, 1948.

See also chapter viii of this yearbook.

the sum is 1 less than when 10 is added to that number); or in carrying in addition or changing (borrowing) in subtraction. If in adding two-place money numbers (e.g., 14 cents + 28 cents) toy coins are available, the children can actually change 10 of the pennies in the sum of the pennies' place for 1 dime and carry it to the dimes' place. In changing in subtraction (e.g., 24 cents – 18 cents) one of the 2 dimes can actually be changed for 10 pennies and these pennies grouped with the 4 original pennies, from which group 8 pennies can then be subtracted.

Carrying and changing should also be demonstrated and clarified for nonmoney numbers. Objects which come in tens or can easily be grouped and fastened in tens are readily available (wooden "sucker" sticks, 2 packages of gum, wooden blocks, individual breakfast-food cartons). The abacus has the added advantage that its individual wires typify the *places* in the decimal system. Simple directions for constructing ten blocks and an inexpensive abacus are given by Spitzer.[11]

In using the abacus when adding involves carrying, one *must* change 10 ones for 1 ten because there are only 10 ones on the ones' wire.

An Active Role for the Learner. When asked by her mother what she did in school, one child replied, "Well, the teacher puts something on the board and we just sit there and wonder what it is." Sitting might reasonably indicate passivity; wondering might indicate activity of a sort. But is it not rather sad to have a child's activity limited to wondering about the mysteries of teacher behavior with little chance of clarification? No doubt, there are large numbers of older children and adults who still "wonder what it is" concerning mysteries of arithmetic that need not have remained mysteries at all.

The work with concrete aids discussed in the preceding sections implies activity in the manipulation, grouping, and regrouping of coins, blocks, beads on a wire, sticks, and other materials. The mere physical activity, however, is not enough. The manipulations must be accomplished by mental activity which, with the physical activity, results in clarification of meanings and organization of experiences in patterns which make sense to the learner.

Teachers are too prone to think of arithmetic teaching as being a matter of telling children facts and dictating procedures, then having the children repeat those facts and procedures as nearly as possible in the same way as that used by the teacher. If the learner has the requisite readiness for the learnings of a new phase of a process in arithmetic, he should be able to take a more active role in the working out of the new phase than he usually gets. If multiplication is a refinement of counting for a special type of

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[11] Spitzer, *op. cit.*, 79–81.

situation, a child should be able to solve a simple multiplication problem without his ever having heard of multiplication. Suppose that 4 children at a table need 2 sheets of paper each. A teacher might use that situation in presenting multiplication, and go immediately to *explaining* 4×2 or 2×4, depending on how the distribution was to take place. She would be giving the pupils a more active role and increasing immeasurably their chances for understanding if she presented the situation and asked the pupils to find out by as many different methods as possible the total number of sheets of paper needed. Some would probably count: 2, 4, 6, 8 or 4, 8. Some would probably add: $2 + 2 + 2 + 2$ or $4 + 4$. Some might use the actual sheets of paper, grouping them in four 2's. The big point is that the chance to work out the solution himself helps the child to see the relationships involved and also helps the teacher to judge his present readiness so that she can do a better job of leading him into the multiplication process.

Enough research studies have been done on the comparative results of different methods of arithmetic instruction in the primary grades to support the statement that methods which lead the children themselves to discover relationships and note generalizations yield superior learning results.[12]

Also in the subsequent application of number learnings in real or practice problem-situations, teachers need to challenge children to go ahead on their own as far as they can. A third-grade group may have a verbal problem like this to solve: A man had to drive 95 miles to get home. His car broke down when he had driven 30 miles. How far did he still have to go to get home? If the children have difficulty, it is so easy for the teacher to start *telling how* and *explaining why,* when she might better suggest, for example, that they draw a line to represent the road and see if they can work out the problem that way. If they have had previous experience with number lines, they will perhaps have little difficulty finding the solution, and their confidence in their ability to solve problems will be bolstered at the same time.

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[12] T. R. McConnell, *Discovery versus Authoritative Identification in the Learning of Children.* University of Iowa Studies in Education, Vol. IX, No. 5, September 15, 1934.

C. L. Thiele, *The Contribution of Generalization to the Learning of the Addition Facts.* Teachers College Contributions to Education, No. 763. New York: Bureau of Publications, Teachers College, Columbia University, 1938.

Esther J. Swenson, "Organization and Generalization as Factors in Learning, Transfer, and Retroactive Inhibition," *Learning Theory in School Situations,* pp. 9–39. University of Minnesota Studies in Education, No. 2. Minneapolis: University of Minnesota Press, 1949.

Concluding Statement

This chapter is by no means a complete discussion of arithmetic for preschool and primary-grade children. Whole areas, e.g., measurement and fractions, have been ignored entirely. Other important considerations, e.g., arithmetic vocabulary, have been mentioned but not developed.[13] The present chapter should be considered, rather, as a discussion of a sampling of arithmetic content and teaching methods which should result in better learning of arithmetic by preschool and primary-grade children.

[13] The reader is especially urged to see chapter ii for the development of significant ideas not treated in this chapter.

CHAPTER V

ARITHMETIC IN THE MIDDLE GRADES

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Critical Nature of Arithmetic in the Middle Grades

Twenty-five years ago, large numbers of children were never graduated from an elementary school because they could not master the arithmetic taught in the middle grades. Retardation increased as children reached the fourth, fifth, and sixth grades. Overage children in large numbers were found in classes with pupils who had progressed at a normal rate. Reading and writing contributed to such retardation, especially in the early grades, but for many boys and girls in the middle grades, as well as for their teachers, arithmetic was the dreaded subject.

While achievement in arithmetic or any other school subject is no longer so important in determining grade promotion as it was a generation ago, the middle grades still remain critical in the teaching of arithmetic. Modifications have been made in the grade placement of certain number concepts and manipulations, and yet there appear to be many secondary-school pupils who did not learn or did not retain the arithmetic studied in the middle grades. Many teachers and students are not satisfied with this state of affairs and continually re-examine the experiences of the fourth, fifth, and sixth grades where complex number manipulation was first encountered.

For most children the arithmetic curriculum of the fourth, fifth, and sixth grades represents the second level or cycle in their number-learning. When they enter the fourth grade, most pupils are in possession of certain basic number concepts and a few of the simpler tools of arithmetic. They have studied simple whole-number meanings and relationships in a systematic manner and have been introduced incidentally to common fractions and certain simple measures.

During the three or four years which follow the third grade, boys and girls are expected to enlarge and refine the number concepts acquired in the lower grades and to add greatly to the skills as well as to the ability to apply arithmetic. As the arithmetic curriculum is now organized, it is assumed that pupils will be schooled in the fundamentals of arithmetic

when they complete the sixth grade or shortly thereafter. Included in the fundamentals are a working knowledge of whole numbers, common and decimal fractions, percentage, part-whole relationships, simple ratios, and the common measures.

The Impact of a New Point of View

Over the years many teachers have accepted or arrived at a point of view which emphasizes the meaningfulness of arithmetic experiences. While this point of view may be helpful in clarifying the goals in arithmetic teaching, it increases rather than decreases the problems of the middle-grade teacher. Teachers are no longer satisfied with a role which consists of identifying appropriate responses, prescribing the material, and striving for mastery by a method of repetition. Instead, they are concerned with broader patterns of behavior. They strive to guide and direct the learning experiences of children in such a manner that thinking dominates the learning process. They seek active pupil participation rather than passive acceptance in the acquisition of the concepts, skills, and abilities of arithmetic. In short, they expect the individual boy or girl to make his own unique contribution to the learning process. Such a program changes radically the part that the teacher as well as the pupil plays in arithmetic instruction.

Whether or not the curriculum now accepted for the middle grades can be taught meaningfully with success cannot be decided on an a priori basis. There are many problems which challenge the research efforts of workers in the field of arithmetic.

Hartung estimated that the total time available for arithmetic instruction in the middle grades was roughly 100 to 125 hours per year.[1] He also suggested that to teach meaningfully the arithmetic course now assigned to the middle grades would require more time than is now allotted. There seems to be no ready answers to questions about arithmetic time allotment or to the manner in which time should be distributed. Likewise, there are no data regarding the relationship of class size to achievement in arithmetic, nor is it known whether a few, some, or all children could learn meaningfully the arithmetic assigned to the middle grades if given the opportunity. As research findings dealing with objectives and methods of teaching accumulate, such questions will be answered, but classroom teachers generally have very little part in determining how soon definitive answers will be available.

However, there are two closely related areas in an arithmetic program

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[1] Maurice L. Hartung, "Major Instructional Problems in Arithmetic in the Middle Grades," *Arithmetic, 1949*, pp. 80–86. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949.

of instruction over which teachers have a large measure of control, namely, in choosing methods of instruction, and in adopting a viewpoint toward arithmetic content. It is the purpose of the writer to deal with selected problems in these areas—problems related to method and content in teaching arithmetic in the middle grades.

The Problem of Method

Conflicting Theories of Arithmetic Instruction

Roughly speaking, classroom teachers choose between two methods of teaching arithmetic which are unlike in many respects. At one extreme teachers deal with the number experiences which children have in their daily life and attempt to make these experiences as meaningful as possible. At the other extreme, teachers bring to their pupils the content of arithmetic as a series of constituent elements arranged by experts according to logic and the relative difficulty of the elements. These elements are to be mastered by pupils in the form in which they are presented by the method of repetition. The content is usually found in a single textbook which children study under the direction of the teacher without any other learning aids. These characteristics identify an instructional method known as the "drill" method. Undesirable as this method of instruction may be, it is still the method which prevails in countless classrooms. The inadequate training of many teachers, poor facilities, an absence of teaching aids, and large classes make a change to more desirable methods difficult, if not impossible, in many instances. It should be noted that the great majority of the teachers now in training or recently trained were themselves taught arithmetic by the "drill" method.

However, forces have operated during the last two decades which have influenced many teachers to discard the "drill" method, at least in part. Particularly during the last fifteen years changes have been wrought in both pre- and in-service teacher education, and textbooks have been revised; in short, much has been said and done to cause teachers to adopt new methods. A new trend in the teaching of arithmetic has been established, even though it does appear that progress has been slow (see chap. vii). There is still a wide gap between theory and practice as they relate to methods of teaching and to the manner in which the content of arithmetic is viewed by teachers.

Assumptions with Regard to Learning in Arithmetic

In the last analysis the assumptions held regarding learning in arithmetic determine choice of methods as well as the manner in which the content of arithmetic is viewed. It is with this thought in mind that assumptions with regard to learning in arithmetic are here stated. These

assumptions are taken as a point of departure for the discussions which follow.

Basic Meanings Stem from Things. Strictly speaking, children should be expected to follow the course of the culture in the acquisition of basic number meanings. They should, as Dewey and McLellan[2] stated, "put number into things." Commenius[3] expressed the same idea in "The Great Didactic" in this manner: "We shall thus tread in the footsteps of the wise men of old, if each of us obtain his knowledge from originals, from things themselves, and from no other source." Dewey added the thought, "Since meanings are not themselves tangible things, they must be anchored to some physical existence."[4] Clearly, then, one of the first problems of this middle-grade teacher is that of anchoring meanings to things.

Chapter ix in this yearbook is devoted entirely to a discussion of learning aids. The chapter includes descriptions of a variety of materials which may be employed as learning aids as well as directions for their effective use. It is sufficient at this point to state that there are many meanings which must be acquired in the middle grades through the use of tangible things.

Symbols Designate Objects and the Manipulation of Objects. If meanings are anchored in things, then it follows that number symbols and terms serve a definite purpose, namely, to designate objects as well as to indicate the ways in which objects are manipulated. If this principle is adhered to, symbolization and meaning must develop together. Children will neither deal with symbols which have no designations nor will they consider meanings which cannot be symbolized.

In the process of symbolizing quantity and quantitative relationships, there occurs an important step, that of translation. The ordinary language of the learner must be translated into the language of mathematical symbols. From the child's own language, teachers may judge whether or not the symbolization is meaningful. It is not likely that the pupil's language will be that found in textbook rules and explanations.

Symbols Are Generalized by the Learners. It is not enough that children acquire ways of recording their number experiences symbolically, but they must also generalize the symbols. Time is wasted when children merely attach symbols to things without taking the further step of noting the presence of what Dewey[5] called "binding principles" among re-

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[2] John Dewey and James A. McLellan, *The Psychology of Number*, p. 61. New York: D. Appleton & Co., 1906.

[3] According to Robert R. Rusk, *The Doctrines of Great Educators*, chap. xviii, sec. 28, p. 103. New York: Macmillan Co., 1918.

[4] John Dewey, *How We Think*, p. 171. Boston: D. C. Heath & Co., 1909.

[5] *Ibid.*, p. 82.

lated number records. These form the structure around which number knowledge is built, and place value is one of the most important of these.

One of the problems of the teacher, then, is to guide, direct, and stimulate children to make number generalizations. For example, in the process of making records of the manipulations necessary to find the sum of certain unlike fractional parts, children should reach the point of recognizing the principle that two or more unlike parts may be measured off into the same-sized parts by the means of a common unit of measure. Once in possession of this generalization, the addition of fractions assumes meaning.

Learners Must Discover Meaning for Themselves. Psychologically speaking, teachers cannot give children meanings. They must be discovered by children themselves. Teachers can do no more than guide, stimulate, create situations, bring about conditions, and, generally speaking, help children gain meaning. If the principle of discovery be sound, there will be small need for "demonstrating" or "explaining" or "telling" or "showing" on the part of teachers. Also, revolutionary changes in the use of textbooks and learning aids and in classroom procedure will occur in many classrooms.

Purpose Should Guide the Acquisition of Meanings. If children are to participate actively in the process of learning arithmetic, there must be purpose in their activities.[6] Learning with purpose partakes of the nature of problem-solving. It is purpose which guides pupils in their organization of ideas; without purposes, pupils are blind followers of the teacher or the textbook. Consider, for example, the difference between learning to make uneven divisions of the type of $3\overline{)34}$ (*a*) by observing a teacher demonstration of the division with learning aids, or (*b*) by being faced with the task of working out a solution of a problem with familiar materials such as dimes and pennies.

Summary

If the principles suggested above are applied to the teaching of arithmetic, the classroom becomes a laboratory in which children do many things. Under ideal conditions, the classroom becomes a workroom in which books, learning aids, pencils, paper, blackboards, and other paraphernalia are used according to the needs of pupils. In it children are active participants in purposeful activities. The activities are geared to the

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[6] *Note:* The importance of creating purposes in the minds of pupils was recognized by the yearbook committee. The committee chose not to deal intensively with the problem because it is a curriculum problem common to the teaching of all of the school subjects. Limitations of space prevented the committee from doing more than touch upon the problem in several of the chapters.

levels of ability possessed by the pupils. The function of the teacher is to guide rather than to direct. This point of view places the content of arithmetic in a new light.

A Point of View toward the Arithmetic Content of the Middle Grades

The literature on the teaching of arithmetic contains many references to two divergent points of view regarding the content of arithmetic. In many quarters, the content of arithmetic still is considered to be a set of larger topics or skills which have been neatly analyzed into constituent elements of difficulty. In this arrangement the separate difficulties are to be mastered in their proper order. As children are led from one difficulty to the next, the skills are thought to acquire meaning and significance. In short, the point of view is taken that parts are put together to make wholes.

In this yearbook an opposite point of view regarding the content of arithmetic has been stated again and again. The assertion has been made that the content of arithmetic is structured around basic meanings. Children first gain some comprehension of certain basic meanings and then refine, expand, enlarge, or extend them. Thus, progress is from general to particular.

In recent times Brownell,[7] Buckingham,[8] Brueckner and Grossnickle,[9] Morton,[10] Spitzer,[11] Van Engen,[12] and Wheat[13] have dealt with meaning in arithmetic. Courses of study and bulletins issued by cities and states during the last decade have likewise featured arithmetic meanings.

When new content is studied, much depends upon the manner in which

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[7] William A. Brownell, *The Development of Children's Number Ideas in the Primary Grades.* Supplementary Educational Monographs, No. 35. Chicago: University of Chicago Press, 1928.

[8] B. R. Buckingham, *Elementary Arithmetic: Its Meaning and Practice.* Boston: Ginn & Co., 1947.

[9] Leo J. Brueckner and Foster E. Grossnickle, *How To Make Arithmetic Meaningful.* New York: John C. Winston Co., 1947.

[10] Robert L. Morton, *Teaching Arithmetic in the Elementary School,* Vol. II. *The Intermediate Grades.* New York: Silver Burdett Co., 1938.

[11] Herbert Spitzer, *The Teaching of Arithmetic.* Boston: Houghton Mifflin Co., 1948.

[12] H. Van Engen, *Developing the Fraction Concept in the Lower Elementary Grades* (April, 1946); *Developing an Understanding of Place Value* (October, 1946); *Teaching Fractions in the Upper Elementary Grades* (December, 1946); *Using a Ten in Subtraction* (July, 1947). Cedar Falls, Iowa: Bureau of Extension Service, Iowa State Teachers College.

[13] Harry G. Wheat, *The Psychology and Teaching of Arithmetic.* Boston: D. C. Heath & Co., 1937.

the early steps are taken. It matters a great deal whether or not children understand the ideas which quantities express and see meanings in the forms by which quantities are symbolized. If children do not form clear-cut ideas about quantities when they are first studied, progress may be greatly impeded.

For various reasons teachers of the middle grades are apt to devote less time to the building of basic meanings than teachers of the lower grades. Consequently they seem to attach less importance to learning aids. It is possible that as the courses of study now are formulated, teachers of middle-grade arithmetic feel forced to slight meanings in order to find time to help children master a large body of skills. This feeling may be enforced further because little has been done in the evaluation of middle-grade arithmetic to indicate the importance of basic meanings.

The writer takes the position that children ultimately will gain greater competence in arithmetic if they have graded experience in arithmetic, moving from basic meanings to the methods of dealing with numbers used in the adult world.

In the discussion which follows, consideration will be given to meanings of (*a*) whole numbers, (*b*) two-place division, (*c*) common and decimal fractions, (*d*) percentage, (*e*) the meaning of ratio, and (*f*) part-whole relationships in middle-grade arithmetic.

It should be noted that dividing the content into topics is only a means of facilitating the procedures. Theoretically, there is unity in the content of arithmetic, and if arithmetic is learned meaningfully, new meanings are built on old ones. A comprehensive treatment of this point of view is found in chapter iii.

Whole-Number Meanings

An examination of arithmetic textbooks widely used in the fourth grade reveals that in the course of one year after leaving the third grade, children are expected to work intelligently with five-place numbers. Examples such as the following may be found in fourth-grade textbooks used widely throughout the United States.

$$\begin{array}{r} 68{,}496 \\ +\underline{26{,}527} \end{array} \qquad \begin{array}{r} 38{,}414 \\ -\underline{13{,}436} \end{array} \qquad \begin{array}{r} 253 \\ \times\underline{278} \end{array} \qquad 6\overline{)13212}$$

Before children enter the fourth grade, they have, in the main, dealt only with numbers comprised of ones, tens, and hundreds. Some knowledge of place value has been acquired. Pupils have applied the principle that tens and hundreds can be added, subtracted, multiplied, and divided in the same manner as numbers from 1 to 9. Number usage has been confined, for the most part, to numbers for which children have perceptual

understanding or to numbers which can be analyzed easily into smaller and better known units. Certain number facts have also been studied and learned to the point of automatic response. During the fourth school year, in most schools, pupils are expected to continue the double job of learning number facts and of computing with them.

Teachers must realize that number-meaning plays an important part in the work with whole numbers in the fourth grade. Positional notation is extended to the left of the familiar tens and hundreds to include the thousands period. Relationships between the first and second periods must be established, i.e., pupils must learn to think of thousands in terms of hundreds, tens, and ones. The decimal nature of the number system must be made quite apparent to boys and girls.

When once the values of larger numbers have been comprehended, children find little difficulty in thinking in terms of the larger values. For example, two thousand of this, five thousand of that, becomes just as understandable as hundreds of like quantities. Thus, the principle that higher decade numbers may be added, subtracted, multiplied, or divided in the same manner as numbers from 1 to 9 may be extended to the thousands. To that end it is recommended that children have considerable experience in adding, subtracting, multiplying, and dividing even thousands, and much of the adding, subtracting, multiplying, and dividing might be done orally. In the haste of teachers to deal with complex numbers comprised of thousands, hundreds, tens, and ones, they may fail to establish the meaning of thousand as a definite number unit.

Not only must the meaning of thousand as a definite number unit be established, but it must also be related to the smaller hundreds, tens, and ones. This is necessary when numbers must be regrouped in order to make many computations. Consider the following examples in which adjustments are required in the solution of the *B* and *D* examples but not in *A* and *C*.

A	*B*	*C*	*D*
6132	5468	7308	5784
×3	×3	$3\overline{)21924}$	$3\overline{)17352}$
18396	16404		

The fact that no reorganization or breakdown of numbers is involved in the solution of the examples *A* and *C* makes for easy solution. The increase in complexity of solutions *B* over *A* and *D* over *C* should serve to indicate some of the difficulties which children experience with whole numbers in the fourth grade. Obviously the problem of the learner is more than learning basic number facts and mechanical methods of computing with numbers. With the help of the teacher, meanings should be established.

Division with More Than One Digit

The experiences of children who learn division by two-place numbers meaningfully differ markedly from those who learn mechanistically. Learned mechanistically, attention becomes focused very early on the difficulty of estimating the quotient figures. As a means of meeting that difficulty, rules usually are introduced. According to Buckingham,[14] the one-rule method (divide the first figure or the first two figures of the dividend by the first figure of the divisor for a trial quotient) works in about 60 per cent of the cases in which the divisor is a two-place number. Children taught mechanistically must spend much time and effort in learning how to resolve the difficulties presented by the other 40 per cent of the cases. Even after time and energy have been spent, it is highly probable that children develop common-sense methods of their own by which they estimate quotients. Taught meaningfully, that is precisely what children would be expected to do anyway.

If division by two-place divisors is viewed meaningfully, children must first realize that division with two-, or more, place numbers serves the same purposes as division by single digit numbers, namely, to part a number or to find the number of times a given number can be measured off in terms of a given unit. To that end, in the study of two-place division, children should make many even divisions with two-place numbers which are multiples of 10 as well as with easily comprehended numbers such as 15 and 25. In preparation for this, multiples of 10, 20, 30, 40, 50, 60, and the like, might be worked out and used in much the same manner as the so-called multiplication tables are at an earlier point in the curriculum. Much of the work might be of the oral type. Divisions would also be recorded, care being taken to record the quotients in their proper places. For example, divisions like the following would be made and recorded.

$$\begin{array}{r}4\\20\overline{)80}\end{array}\qquad\begin{array}{r}6\\30\overline{)180}\end{array}\qquad\begin{array}{r}7\\60\overline{)420}\end{array}\qquad\begin{array}{r}9\\500\overline{)4500}\end{array}$$

Certainly stating that 4500 can be separated into 9 equal parts, each of which is 500 in size, or that 420 contains 60 evenly 7 times, should not be much more difficult than indicating the division of 45 by 5 or 42 by 6 provided that 60 and 500 represent units familiar to the pupils and the relationships between the divisors and dividends can be sensed without much difficulty. Outside the classroom it is sensible to make divisions with two-, or more, place divisors without the use of algorisms.

In order to develop deeper insight in two-place division, it would seem advisable to help children acquire the ability to relate division by two-

[14] Buckingham, *op. cit.*, p. 199.

place numbers not multiples of 10 to multiples of 10. For example, division by 19 and 21, 29 and 31, 39 and 41 can be related to division by 20, 30, and 40, respectively. In other words, division by the method of inspection can be introduced. Children can be led to discover that a particular division like $21\overline{)84}$ can be made by recalling $4 \times 20 = 80$ and then relating 4×21 to 4×20. Likewise, when dividing 87 by 29, reference might logically be made to $3 \times 30 = 90$ to be followed by the adjustment for 3 times the number one smaller than 30.[15]

A few children experience little difficulty in working out division by the method of inspection with divisors two and even three more or less than apparent divisors. Given the opportunity, some children become skilled in solving division problems by the method of inspection. Smith[16] has proposed even a wider use of informal methods of making two-place divisions than has been made in the foregoing paragraphs.

Evidence from research studies is not at hand to indicate that children have less difficulty in learning two-place division if the introduction of the algorism is preceded by informal experiences of the kind described in the foregoing paragraphs. However, it seems logical that if pupils first become thoroughly familiar with two-place division by making many even

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[15] This illustration may serve to indicate that mathematical learning builds up from grade to grade. With proper guidance when the multiplication combinations are studied in the third grade, children may discover how products are related to one another in accordance with relationships which exist between multipliers and between multiplicands. Common among these relationships, which might be generalized, are those indicated in the examples which follow:

A	B	C
$2 \times 4 = 8$	$2 \times 4 = 8$	$3 \times 7 = 21$
$2 \times 8 = 16$	$4 \times 4 = 16$	$4 \times 7 = 28$
$3 \times 7 = 21$	$4 \times 6 = 24$	$4 \times 6 = 24$
$2 \times 7 = 14$	$4 \times 7 = 28$	$5 \times 6 = 30$

The ability to use these generalizations in rounding off numbers and making adjustments cannot be acquired in a short time. As early as the second grade children can be encouraged to solve problems by that method. Properly challenged, many second- and third-grade pupils can learn to add amounts like 20 and 19, 15 and 16, and 25 and 24, and to make subtractions like $30 - 14$, $40 - 21$, and $50 - 24$, mentally, by relating the additions and subtractions to number situations for which they have ready answers. Likewise, in the study of multiplication by single digits, problems like 4×52 and 8×99 may be solved by the method of inspection. Thus an application of the inspection method to two-place division may become an extension of a technique with which children are familiar. This illustrates the fallacy of trying to consider any part of the arithmetic curriculum without reference to what has preceded it. This illustration also should serve to indicate that the development of meaning and insight must be sought in all grades.

[16] Rolland R. Smith, "Meaningful Division," *Mathematics Teacher*, XLIII (January 1950), 12.

divisions mentally with simple two-place divisors, and then find quotients by inspection, that the step to using an algorism should be taken easily. The algorism becomes another method of doing something which children can already do and understand.

Common and Decimal Fractions

Common-Fraction Meanings. Of the three meanings which fractions are used to show, namely, (*a*) a part of a whole or of a group, (*b*) an indicated division, and (*c*) a ratio, the first finds the widest application in the lives of boys and girls. It is experienced by children before they think very sharply about the other two. The three meanings are so distinctly different that each requires the attention of the teacher.

a) When a study of fractions is undertaken, the ideas which fractions express and the form of fractional notation must go hand in hand. For example, $\frac{2}{3}$ of something means that either a single object or a group of objects has been divided into three equal parts, and that two of them are being considered. The denominator 3 tells the relative size of each part and the numerator 2 the number of parts. Fractions deal with two things: number of parts and size of part. This double meaning of numbers represents a step beyond the single meaning given to integer numbers. In order to familiarize children with the meaning of this new notation before they start to compute with fractions, they should engage in many experiences of measuring, parting, dividing, and the like, and should make notations of their activities. Finally the application of taking a part of a whole or of a group should be extended to abstract numbers.

b) At a much later time experiences with objects or representations of objects should lead to an understanding of fractions as indicated divisions. For example, $\frac{2}{3}$ is not only two-thirds of one thing but it is also one-third of two. It is a simple matter for children to find that one-third of 2 dozen objects, of the inches in 2 feet, of the pencils in 2 packages of equal number, etc., is the same as two-thirds of one dozen objects, two-thirds of the inches in 1 foot, and so on. In this instance $\frac{2}{3}$ means 2 divided by 3. The line between the numerator and the denominator means a division. This meaning provides the authority for dividing the numerator by the denominator when common fractions are changed to decimal fractions.

c) The ratio meaning of fractions has its beginnings in experiences to which expressions such as "three out of four," "nine out of ten," and the like, may be connected. For example, in the scoring of pupils' arithmetic tests, fractions such as $\frac{9}{10}$, $\frac{15}{20}$, and $\frac{22}{23}$ may be used to indicate the number of problems correct out of a total number. Nine out of ten used in this manner may mean not only that nine-tenths of the problems were correct but also that nine out of ten were correct. On the abstract level every

fraction expresses a ratio between the number of parts and the unit by which something is measured. It is this meaning which is basic to an understanding of percentage.

Two courses are open to teachers after children have learned something about the meanings of a common fraction. In some classrooms the teacher chooses to lead children immediately into computations with fractions without adequate preparation. In so doing, children may be shown how to reduce fractions and how to find equivalent fractions by mechanistic methods. The most common method by which children are taught to find equivalent denominators is that of multiplying the larger or the largest denominator by two, and then by succeeding numbers until the first number is found into which the denominators in question can be evenly divided. Fractions are reduced by the simple expedient of dividing numerator and denominator by the same number until a fraction has the smallest numbers it can have in the numerator and denominator. In themselves, these methods are not bad, but teaching is faulty when children are shown devices before they are ready for them—before the devices have real meaning.

Even among those who subscribe to the idea that arithmetic should be meaningful to children, there is little agreement on how far the development of meaning should be carried. Johnson[17] questioned the extent to which meanings could be developed in a recent article entitled, "What Do We Mean by Meaning in Arithmetic?" Textbook authors are by no means in agreement on the kinds of experiences children should have before they are expected to compute with numbers on the abstract level.

It seems both logical and consistent to hold that children should not be expected to make computations on the abstract level until the operations can be explained and demonstrated in situations that are real, that have meaning, to them. For example, in the teaching of common fractions children should be able to reduce simple fractions and find equivalent fractions with objects, or pictorially, before they learn any of the methods which adults commonly employ. Van Engen described the role of the teacher in this connection very well in these words:

> The teacher must work on a level at which it is actually possible to perform the operation and then symbolize the operation by means of the usual arithmetic symbols. When the operation and the symbols are firmly united in the minds of the pupils, then the teacher can generalize the symbols by edging over into the area in which it is inconvenient or even impossible to perform the operation.[18]

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[17] J. T. Johnson, "What Do We Mean by Meaning in Arithmetic?" *Mathematics Teacher*, XLI (December, 1948), 362.

[18] H. Van Engen, "Place Value and the Number System," *Arithmetic 1947*, p. 59. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947.

The teacher must provide many experiences in which the child unites the operations of reducing fractions and finding equivalent fractions with the symbols which are conventionally used.

Analysis and Synthesis of Wholes and Parts of Wholes. As a first step toward developing meaning in the reduction of fractions and finding equivalent fractions, middle-grade pupils may engage in activities which involve the division of whole and parts of wholes into equal parts and putting them together again. Normally these activities will be related to measures of length, time, and volume and to the methods which are employed in dealing with parts smaller than wholes in the affairs of everyday life. Wholes may be successively divided into 2, 4, 8, and 16 equal parts; into 3, 6, 12, and 24; and into 5, 10, and 20 equal parts. The results of the divisions will be recorded as $1 = \frac{2}{2} = \frac{4}{4} = \frac{8}{8} = \frac{16}{16}$, etc. Wholes may also be divided successively into 2 parts, 3 parts, 4 parts, etc., and the divisions symbolized as $1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4}$, etc. Equal parts may also be combined in this fashion: $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$; $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1\frac{1}{4}$; etc. In this manner children may comprehend the relationships between wholes and parts and between parts themselves. They may, for example, become aware of the relationships between number of parts and the size of a part as represented by the relationships between $\frac{2}{2}$ and $\frac{8}{8}$, between $\frac{3}{3}$ and $\frac{9}{9}$, $\frac{4}{4}$ and $\frac{12}{4}$.

If reducing and finding equivalent fractions is to have real meaning, parts of wholes should also be analyzed and synthesized. To that end children should divide parts of wholes such as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ into smaller parts by successive halving of parts. The results would be recorded as $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$, $\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$, $\frac{1}{5} = \frac{2}{10} = \frac{4}{20}$, etc. Parts such as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ would also be divided into 2 equal parts, 3 equal parts, 4 equal parts, etc., as wholes were. Records of the divisions would be made in the form of $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$; $\frac{1}{5} = \frac{2}{10} = \frac{3}{15} = \frac{4}{20}$; etc. Experience in analyzing and synthesizing fractions also may be extended to fractions like $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{2}{5}$, all of which contain more than one part. Actual parts or graphic representations of them may be divided and combined in the same manner as unit fractions and wholes.

With proper guidance, children may thus learn to reduce many simple fractions and find their equivalents on a meaningful level. Accordingly, the key to all later work with fractions may be discovered, namely, that multiplying or dividing both terms of a fraction by the same number does not change its value. For example, children may discover that if the size of a given fraction is made one-half or one-third as small, then the number of parts is doubled or trebled, whichever the case may be. Likewise, if the number of parts in a fraction is reduced by one-half or one-third, then the size of the parts will become twice or three times as large. For a more detailed account of the basic rule of fractions, the reader is referred to an

account by Buckingham.[19] Activities which may be conducted in the classroom have been described by Sauble.[20]

If the recommendations made by Buckingham and Sauble, as well as by others, were made a part of the course of study in middle-grade arithmetic, much more time would be spent in developing basic meanings of fractions than now is the case in most classrooms. Many more learning aids would be used than are now employed. Consequently, there would be less time available for the great amount of computational work that now comprises a large part of the middle-grade work with fractions. But there would be greater assurance that meaning and understanding was preceding the teaching of skill in computation on the abstract level.

The Fraction Multiplier. In the experiences of children, the fraction multiplier serves one purpose, namely, to facilitate multiplication by mixed numbers. At no time are numbers multiplied by common fractions such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{8}$, and the like. The language of arithmetic makes no provision for employing common-fraction multipliers in the same manner as whole-number multipliers. For example, we do not say that a distance is "$\frac{1}{2}$ times" as great as a known distance, nor one weight is "$\frac{3}{4}$ times" another weight. Instead, when it becomes necessary to relate multiplicands to common-fraction multipliers, the multiplicands are parted—one distance is "$\frac{1}{2}$ of" another, and one weight is "$\frac{3}{4}$ of" another weight. Strangely, though, we do not indicate that one weight is "$2\frac{1}{2}$ of" another weight. The inconsistency of parting when actually the process is one of multiplication has been so well established that there is slight prospect of removing the term "of," the cause of the inconsistency, from the language of arithmetic. Children will continue to part numbers rather than employ common-fraction multipliers in the manner which they do integer multipliers.

The need for understanding the unique function of a fraction multiplier will arise first when numbers must be multiplied by mixed numbers. At that time meaning can be established by relating multiplication by fraction multipliers to multiplication by 1. This relationship may be called to the attention of children through exercises of the following kind:

$4 \times 8 = 32$	$6 \times 6 = 36$	$\frac{1}{2} \times 2 = 1$	$\frac{1}{4} \times 4 =$
$2 \times 8 = 16$	$3 \times 6 = 18$	$\frac{1}{2} \times 1 = ?$	$\frac{2}{4} \times 4 =$
$1 \times 8 = 8$	$1 \times 6 = 6$	$\frac{1}{2} \times \frac{1}{2} = ?$	$\frac{3}{4} \times 4 =$
$\frac{1}{2} \times 8 = ?$	$\frac{1}{3} \times 6 = ?$	$\frac{1}{2} \times \frac{1}{4} = ?$	$\frac{1}{3} \times 3 =$
$\frac{1}{4} \times 8 = ?$	$\frac{1}{6} \times 6 = ?$	$\frac{1}{2} \times \frac{1}{8} = ?$	$\frac{2}{3} \times 3 =$
$\frac{1}{8} \times 8 = ?$	$\frac{1}{12} \times 6 = ?$		

[19] Buckingham, *op. cit.*, p. 250.

[20] Irene Sauble, "Teaching Fractions, Decimals, and Per Cent: Practical Applications," *Arithmetic 1947*, p. 33. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947.

Having noted the effect of whole number multipliers on a given multiplicand in relationship to 1 and to other numbers, children will be in a position to relate fraction multipliers to 1. On this basis $\frac{1}{2} \times 8$ cannot possibly be 16, as children taught mechanistically frequently think. It must be one-half as much as 1×8. By the same token, the product of $\frac{1}{4} \times 8$ must be one-fourth as much as 1×8, or one-half as much as $\frac{1}{2} \times 8$. When once children see sense in obtaining products with fraction multipliers, multiplication with mixed numbers assumes new meaning. With some practice they gain the ability to judge the reasonableness of products.

Again attention is called to interrelatedness in number-learning. Multiplication by fraction multipliers is not an ability that can be acquired without reference to multiplication by whole numbers, nor can parting numbers be made the basis for an understanding of multiplication by fraction multipliers. They are two distinctly different operations, and one should not be confused with the other.

Dividing with a Fraction Divisor. Most of the present-day arithmetic textbooks are written with the thought of giving meaning to the division of fractions by objectifying the process. Not only are lengths separated into a given number of smaller units, but number of parts is also found by successive subtractions. Following this the common-denominator method is presented in most textbooks as a way of solving problems symbolically. Then the reciprocal or the inversion method also is presented, not as a short-cut for the common-denominator method but as a distinctly different method with a meaning of its own. Thus, children are introduced to two methods of dividing with fractions without being made aware of the fact that one may be related to the other. Sample solutions of corresponding examples by each method follow.

Common-Denominator Method:

a) $8 \div \frac{1}{4} = \frac{32}{4} \div \frac{1}{4} = \frac{32}{1} = 32$

b) $3\frac{1}{2} \div \frac{1}{4} = \frac{7}{2} \div \frac{1}{4} = \frac{14}{4} \div \frac{1}{4} = \frac{14}{1} = 14$

c) $8 \div \frac{2}{3} = \frac{24}{3} \div \frac{2}{3} = \frac{24}{2} = 12$

d) $3\frac{1}{2} \div \frac{2}{3} = \frac{7}{2} \div \frac{2}{3} = \frac{21}{6} \div \frac{4}{6} = \frac{21}{4} = 5\frac{1}{4}$

e) $3\frac{1}{3} \div 5\frac{1}{2} = \frac{10}{3} \div \frac{11}{2} = \frac{20}{6} \div \frac{33}{6} = \frac{20}{33}$

Reciprocal Method:

a) $8 \div \frac{1}{4} = \frac{8}{1} \times \frac{4}{1} = \frac{32}{1} = 32$

b) $3\frac{1}{2} \div \frac{1}{4} = \frac{7}{2} \times \frac{4}{1} = \frac{28}{2} = 14$

c) $8 \div \frac{2}{3} = 8 \times 1\frac{1}{2} = 8 \times \frac{3}{2} = \frac{24}{2} = 12$

d) $3\frac{1}{2} \div \frac{2}{3} = 3\frac{1}{2} \times \frac{3}{2} = \frac{7}{2} \times \frac{3}{2} = \frac{21}{4} = 5\frac{1}{4}$

e) $3\frac{1}{3} \div 5\frac{1}{2} = \frac{10}{3} \div \frac{11}{2} = \frac{10}{3} \times \frac{2}{11} = \frac{20}{33}$

An examination of examples solved by the common-denominator and the reciprocal methods will indicate that the two methods are in no sense related in meaning. For example, in the solution of $3\frac{1}{2} \div \frac{2}{3}$ by the com-

mon-denominator method, the problem becomes one of determining how many $\frac{4}{6}$ there are in $\frac{21}{6}$ after $3\frac{1}{2}$ and $\frac{2}{3}$ have been changed to sixths. The solution employs the division concept. If the same problem is solved by the reciprocal method, the dividend $3\frac{1}{2}$ is multiplied by $\frac{3}{2}$, the reciprocal of the divisor $\frac{2}{3}$. This is justified on the grounds that, when divisions are made, a number is sought which when multiplied by the divisor will give the dividend. To illustrate, in the division of 12 by 4, the dividend will be a number which when multiplied by 4 will yield 12. $\frac{1}{4}$, the reciprocal of 4, times 12 produces that number, 3. Only indirectly is the reciprocal method related to the division idea.

In a program of arithmetic instruction in which meanings assume great importance to the learner, it would seem inadvisable to acquaint children with two methods as unlike as the common-denominator and the reciprocal methods for solving problems in which fraction divisors are employed. Teachers who have attempted to acquaint children with both the common-denominator and the reciprocal method report that children are greatly confused by this experience. It seems more logical that the reciprocal method should not be related to the solution of problems involving division by fraction divisors. The idea that any number may be divided by another by multiplying the number by the reciprocal of the divisor is of such importance in arithmetic that it should be treated as a special topic rather than introduced for the purpose of justifying the inversion method. In fact, it is not necessary to introduce the reciprocal method at all to lead pupils to a method shorter than the common-denominator method.

If in using the common-denominator method, the common denominator is always assumed to be the product of the denominators of the dividend and the fraction divisor, then the numerators may be obtained by cross multiplication. Buckingham[21] reminds us that at one time the cross multiplication was widely used "whereby the numerator of the dividend was multiplied by the denominator of the divisor for the numerator of the quotient, and the denominator of the dividend by the numerator of the divisor for the denominator of the quotient." Note in the illustration below how much the task of finding the quotient fraction is shortened by the cross-multiplication method.

$\frac{7}{8} \div \frac{2}{3} = \frac{21}{24} \div \frac{16}{24} = \frac{21}{16}$ The quotient fraction.

$\frac{7}{8} \div \frac{2}{3} = \frac{21}{16}$ The quotient fraction.

$4 \div 1\frac{3}{4} = \frac{4}{1} \div \frac{7}{4} = \frac{16}{4} \div \frac{7}{4} = \frac{16}{7}$

$4 \div 1\frac{3}{4} = \frac{4}{1} \div \frac{7}{4} = \frac{16}{7}$

$\frac{3}{4} \div \frac{7}{8} = \frac{24}{32} \div \frac{28}{32} = \frac{24}{28}$

$\frac{3}{4} \div \frac{7}{8} = \frac{24}{28}$

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[21] Buckingham, *op. cit.*, p. 276.

Doubt in deciding which cross multiplication yields the numerator and which the denominator of the quotient fraction may be removed by inverting the divisor. Approached in this manner, inversion becomes an expedient way of obtaining quotient fractions which in turn have meaning.

It matters not whether all children gain the ability to use the inversion method in division involving fraction divisors. It is more important that they understand the method which they do use. For the same reasons that some children must continue to use crutches through the grades when solving certain subtraction problems, children will not be able to move from the common-denominator method to the level of inverting fraction divisors. In guiding children, teachers must be able to judge the capacities of children to see meaning in what they study.

Meaning of Decimal Fractions. The problem of uniting an operation and its symbols is the central issue in teaching the meaning of decimal fractions. Decimal fractions do not differ in meaning from common fractions. They designate parts and ratios and indicate division just as common fractions do. The operations with decimal fractions have the same meanings as the corresponding operations with common fractions. The chief differences between common fractions and decimal fractions are that common-fraction denominators may be any number, but decimal-fraction denominators are always powers of ten. Furthermore, decimal fraction denominators are implied rather than stated. Even though the chief difference between common fractions and decimal fractions is in the manner in which the two kinds of fractions are written, that difference creates many problems. This discussion is concerned with those which involve meanings and understandings.

In the early stages of teaching decimal fractions, it is common practice to provide children with experiences in direct measurement by means of decimal units. The extent to which children need these experiences depends largely upon how well they understood similar measurements with common-fraction units. The chief value of measuring exercises should be to give children a clearer conception of decimal units and of relationships among them.

Following the introductory experiences with decimal units, two courses are open to teachers. If one is taken, the notation of decimal fractions will be related to place value at the outset. The number system will be extended to the right of the one's place in the same manner as it earlier should have been extended to the left of the one's place. If the other path is pursued, decimal fractions will be directly related to common fractions and common-fraction operations at every step of the way. In so doing, importance will be given to common fractions with denominators of 10 and powers of 10. Only incidentally, if at all, will decimal-fraction meanings be obtained from an application of the principle of place value.

Of the two developments, that of relating decimal fractions to common fractions is found almost exclusively in present-day textbooks and courses of study. Few children in American schools today learn that the one's place is the center of the number system. Even though they learn that .1, .01, and .001 mean one tenth, one hundredth, and one thousandth, respectively, they do not become aware of the basic fact that each decimal fraction is some part of 1. As a consequence, they do not possess the means of determining the correct value of decimal fractions if the memory of a chart or a diagram or some device fails them. To them arithmetic is not a system of related ideas, but compartmentalized bits of knowledge.

Addition and Subtraction with Decimal Fractions. Only recently have textbooks appeared in which may be found meaningful explanations for the addition and subtraction with decimal fractions. Without explanation, children have been told that decimal points must be placed in a straight line before columns are added or subtracted. This expediency has not prevented some pupils from becoming confused when they have added and subtracted decimal fractions of unlike denominations. By the simple device of changing decimal fractions to common denominators, as is done with common fractions before adding or subtracting, a little-understood rule has been clarified. In the example which follows, decimal fractions are added with and without changing them to common denominators.

A		*B*	
.24	Hundredths	.240	Thousandths
.366	Thousandths	.366	"
+.2	Tenths	.200	
.806	Thousandths	.806	

This example again illustrates interrelatedness in arithmetic and the possibility of building new concepts out of experiences with simpler concepts.

Multiplication with Decimal-Fraction Multipliers. The issue when pupils find a need for multiplication with decimal-fraction multipliers is the same as when they are called upon to employ common-fraction multipliers. In both cases the products are smaller than the multiplicands, i.e., if the multipliers are proper fractions. This is an important concept which children must gain. Without it they lack the ability to judge the reasonableness of many products.

Children are led to see the logic of obtaining products smaller than the multiplicands in two ways. One way is to change the order of the multiplier and multiplicand for problems in which the multiplier is an integer and the multiplicand a proper fraction. For example, changes are made in the order of multiplier and multiplicand like the following:

$8 \times .5 = 4.0$ changed to $.5 \times 8 = 4.0$
$8 \times .05 = .40$ changed to $.05 \times 8 = .40$

The other explanation takes into account the function which the multiplier serves, that of indicating how the multiplicand is to be treated. If reliance is placed upon this explanation, multiplication by proper-fraction multipliers, be they common or decimal, is related to multiplication by 1. To illustrate, the product of .5 × 8 will be one-half as large as the product of 1 × 8 because the multiplier .5 is one-half of 1. It is assumed that in the study of whole-number multiplication children become aware of the fact that changes in multipliers produce corresponding changes in products if the multiplicand remains constant. To the end that this concept may be extended to multiplication involving proper-fraction multipliers, pupils may work out and analyze a series of multiplication relationships such as the following:

A	*B*	*C*
1 × 8 = 8	3 × 80 = 240	3 × 400 = 1200
10 × 8 = 80	2 × 80 = 160	2 × 400 = 800
100 × 8 = 800	1 × 80 = 80	1 × 400 = 400
1000 × 8 = 8000	.1 × 80 = ?	.01 × 400 = ?
100 × 8 = 800	.2 × 80 = ?	.02 × 400 = ?
10 × 8 = 80	.3 × 80 = ?	.03 × 400 = ?
1 × 8 = 8	.4 × 80 = ?	.04 × 400 = ?

D	*E*
1 × 10 = 10	1 × 100 = 100
.1 × 10 = ?	.01 × 100 = ?
.1 × 9 = ?	.01 × 90 = ?
.1 × 8 = ?	.01 × 80 = ?
.1 × 7 = ?	.01 × 70 = ?
.1 × 6 = ?	.01 × 60 = ?
.1 × 5 = ?	.01 × 50 = ?

The net result of experiences such as those suggested should be that pupils will become aware of the effects produced when numbers are multiplied by 100, 10, 1, .1, or .01. Awareness of the effect of multiplying by numbers directly related to 1 should enable children to discover the rules for placing decimal points in products and, at the same time, should lead pupils to refine their concepts of place value.

Of the two methods described above, preference is expressed for the second because, in learning to multiply by fraction multipliers by the method of relating fraction multipliers to 1, previous learning is reorganized in a direct manner, i.e., children are given an opportunity of deducing new concepts from old ones. The trick of reversing multiplier and multiplicand may be classified as a verification of the process rather than as an explanation, in contrast with relating proper-fraction multiplication to multiplication by 1. The fact that a reversal of multiplier and multiplicand does not change the product does not explain the part that the multiplier plays; it merely gives proof that the order can be changed

without altering the results. It is highly important that the effect of multiplying with proper-fraction multipliers be recognized when children multiply with mixed decimals. Aware of the effect, children are not apt to obtain a product such as 500 for an example like 2.5×20 because they will realize the part that the .5 plays in the process.

In passing, it would seem in order to comment on the kind of learning experience through which it is thought that children can make deductions regarding multiplication by proper decimal-fraction multipliers. The learning experience which was proposed involves only the analysis of number records like $2 \times 80 = 160$, $1 \times 80 = 80$, $.1 \times 80 = 8$; and $2 \times 80 = 160$, $1 \times 80 = 80$, $.01 \times 80 = .8$. There is in the activity a complete absence of learning aids, in the sense that learning aids commonly are viewed. The problem of the learner is that of reorganizing ideas entirely on the abstract level. In the teaching of arithmetic there are many instances when pupils should be able to build new concepts without the immediate benefit of concrete learning aids.

Division with Decimal Fractions. If children understand certain principles which relate to place value, only one new principle must be understood to divide decimal-fraction numbers successfully. The new principle is that divisions can be made more conveniently if divisors first are transformed into whole numbers. Consider the following examples which contain a few of the possible variants:

$.6\overline{)3.}$	$= 6\overline{)30}$	$1.6\overline{)3.2}$	$= 16\overline{)32}$
$.06\overline{)3.}$	$= 6\overline{)300.}$	$1.6\overline{)32}$	$= 16\overline{)320}$
$.6\overline{).3}$	$= 6\overline{)3.}$	$1.6\overline{).32}$	$= 16\overline{)3.2}$
$.6\overline{).03}$	$= 6\overline{).3}$	$1.6\overline{).0032}$	$= 16\overline{).032}$

Note that in each case after the transformation has been made only a knowledge of place value as it applies to division is required to complete the problem. However, that is an oversimplification of the process because a number of concepts must operate together to make possible the transformation, the division, and the interpretation of the results. First, the learner must think of the divisors as denominators and the dividends as numerators of common fractions, i.e., $.6\overline{)30} = \frac{30}{.6}$. He must recall the fact that the value of a fraction is unchanged if the numerator and denominator are multiplied by the same number. Too, he must know that multiplying decimal fractions and whole numbers by 10 and powers of 10 has the effect of moving decimal points or of adding zeros. Finally, he must employ his knowledge of place value when he divides and interprets the results. The fact that as children progress in arithmetic they are required to combine skills and abilities in the manner described above makes middle-grade arithmetic a difficult subject for many children.

However, there is an important step that frequently must be taken before the idea of dividing decimal fractions by whole numbers is introduced. The teacher should provide experiences which will give assurance that children can divide smaller numbers by larger ones. As a rule, courses of study and textbooks are so organized that children find purposes for dividing integers by larger integers before they find a need for dividing with decimal-fraction divisors. This is frequently necessary when common fractions are changed to decimal fractions in order to make comparisons or to indicate with a decimal fraction what part one number is of another.

To divide a smaller number by a larger one meaningfully, a knowledge of the decimal nature of the number system is required. For example, in order to divide 4 by 6, the 4 must be analyzed to mean 40 tenths. Likewise, when 2 is divided by 25, it is not 2 as such that is divided, but 200 hundredths. Without this knowledge, the division with decimal divisors must be learned more or less by rote.

If children have the readiness for division by decimal fractions, which has been outlined, only one step need be taken to help them discover the advantage of changing the decimal-fraction divisor to a whole number before dividing. The step can be taken by focusing the attention of pupils on examples such as $7\frac{1}{2} \div 1\frac{1}{2}$, $15 \div 2\frac{1}{2}$, and $20 \div 3\frac{1}{3}$, the solutions of which can be made easier by simple transformations. For example, $1\frac{1}{2}\overline{)7\frac{1}{2}}$ may be changed to $3\overline{)15}$, $2\frac{1}{2}\overline{)15}$ to $5\overline{)30}$, and $3\frac{1}{3}\overline{)20}$ to $10\overline{)60}$ by multiplying divisors and dividends by the same number to make the divisors whole numbers. Properly guided, pupils will discover that it is also expedient to change decimal-fraction divisors to whole numbers. After pupils have come to that realization, the conventional rules for moving decimal points in the divisor and dividend and placing the decimal point in the quotient may be deduced. As in all meaningful teaching, greater importance must be attached to an understanding of how and why numbers are operated than to the learning of rules and devices.

An analysis of the operations with decimal fractions which involve the division process should serve to indicate the complexity of the teaching problem. Growth in number-thinking depends upon the ability of pupils to make new interpretations of old ideas and to organize learning in new ways. It would seem then, in the teaching of arithmetic, that time must be taken to build meaning into the structure.

Percentage

A major problem in the teaching of percentage is that of determining how much meaning should be given to percentage. Superficially, percentage is another way of writing hundredths. It is a decimal fraction written in words or designated by the percentage sign. Ordinarily, in the teaching

of percentage the per cent concept is objectified with charts representing a rectangle divided into one hundred parts, or by a length divided into one hundred units.

Following this introduction to the per cent concept, children are taught (*a*) to find a per cent of a number by changing the stated per cent to hundredths and multiplying, (*b*) to determine what per cent one number is of another by finding first what part one number is of another in terms of hundredths and then changing the hundredths to per cent, (*c*) to find a number when its per cent of the whole is known either by first determining 1 per cent and then 100 per cent, or by dividing the stated number by its per cent, expressed decimally. Children can study percentage on this level without becoming aware of its true meaning.

In chapter ix of this volume the authors have indicated how percentage may be learned in a more meaningful way. As explained in that chapter, per cent means "out of" or "by the hundred." The suggestion is made that pupils be faced with problems involving comparisons which can best be made by changing ratio fractions to common denominators of 100. Thereby 100 is established as a convenient base to which fractions may be changed for the purpose of making comparisons. Children who are familiar with frequently used ratios such as "2 out of 3," "4 out of every 5," "9 for each 10," "40 out of 50," and "90 out of every 100" will appreciate the value of translating all ratios into hundredths.

When once the concept of per cent has been partially understood by pupils, teachers are faced with the problem of deciding how soon the algorisms of percentage should be learned by their pupils. This is a recurring problem in the teaching of arithmetic. In order to complete course-of-study and textbook assignments, most middle-grade teachers feel forced to lead children very quickly into the algorisms of percentage. Difficulties arise when this is done before they have gained clear, well-organized, accurate ideas about percentage. The result is that one meaning does not follow another, but rather that a gap is established between the first meanings of percentage and the algorisms, and the algorisms are learned mechanistically.

If a sound knowledge of percentage is to be built, it would seem advisable to delay the introduction of algorisms. Not only might pupils continue to find what per cent one number is of another by the method of changing ratios to hundredths but they might also find a per cent of a number without the aid of an algorism. For example, if 40 per cent is recognized as 40 "out of a hundred" or "out of every hundred," then deductions such as the following may be made:

> 40 per cent of 200 is 40 plus 40. (40 for each 100)
> 40 per cent of 250 is 40 plus 40 plus 20. (20 for the half hundred)

Per cents of certain numbers which are less than 100 may also be deduced. For example, in finding 40 per cent of 75, pupils might reason as follows: 40 per cent of 100 is 40, therefore 40 per cent of 75, which is $\frac{3}{4}$ of 100, will be three-fourths as much as 40, or 30. After experiences of this type, finding the per cent of a number is likely to be more meaningful.

The same sort of simple procedure may be employed by pupils in their first attempts to find a number of which a part and its per cent are known. This can be done when numbers previously have been found whose relative size has been expressed either as a common or a decimal fraction. Slight reorganization of learning should be required in moving from common fractions to decimal fractions to percentage problems, as represented in the following series:

200 = $\frac{2}{5}$ of what number?
200 = .40 " " "
200 = 40% "

The question, "200 is 40 per cent of what number?" may be answered meaningfully by children in some such manner as: "200 is not all of the parts. It is 40 out of every 100. The whole or 100 per cent would be $2\frac{1}{2}$ times as much as 40 per cent of the number (200), or 500." Another method to be deduced by children might be that of finding 1 per cent and then the whole number. That is precisely what is done when pupils reason, "If 2 out of 5 equal parts is 200, then one part is $\frac{1}{2}$ of 200, or 100, and the whole is 5 times 100, or 500."

Once more attention is called to the basic principle of instruction which has been stressed throughout this discussion, namely, that basic meanings in arithmetic should be thoroughly understood before children are expected to reorganize learning on higher levels. Even in the absence of research evidence to prove the validity of this principle, it would seem proper to recommend that children should not use percentage algorisms until they have demonstrated their ability to solve the three types of percentage problems without them.

The Meaning of Ratio

Behind the statement that ratio is another way of comparing numbers are many ideas. The concept of ratio is an important one. The possibilities of its application in the affairs of everyday life are limitless. The task of helping children gain a concept of ratio is the responsibility of the middle-grade teachers rather than of those in higher grades. In the middle grades the ratio concept may be related to common- and decimal-fraction concepts and thus developed in a logical manner.

Two criticisms may be made of the manner in which ratio commonly

is taught at the present time. One is that children are not given a concrete setting for the ratio concept, and the other that the applications of ratio are too limited to give the concept meaning. Teachers of the higher grades assume that their pupils understand what ratio means and therefore begin their teaching of ratio on the abstract level. The application of the ratio idea in courses of study and current textbooks is limited almost entirely to finding ratios after the numbers to be compared are known.

In the affairs of everyday life people frequently are required to find the numbers involved in a comparison, knowing only the total number and the ratio between the two numbers which comprise the total. For example, if in an election 1,400,000 votes are cast and the ratio between the winning vote and the losing vote is announced as 5 to 2, $2\frac{1}{2}$, or $\frac{2}{5}$, the number of votes polled by winner and loser may be found. Likewise, when the chances of success are stated as 1 out of a hundred, the ratio of success to failure is 1 to 99. With this knowledge, the number of successes out of a total number such as 250,000 may be determined. Few children are confronted with problems of this type in their mathematics classes until they study algebra. Suggestions for developing the ability to solve ratio problems of the type described are presented in the following paragraphs.

The early applications of the ratio concept which children make are usually of the 2 to 1, 3 for 1, 4 to 1, etc., variety. Children separate numbers of articles such as picture cards, coins, and marbles on the 2 to 1, 3 for 1, and 4 to 1 basis. On the basis of such experiences, they become aware of the fact that in each separation there are two parts, the 1 and the 2, the 3, or the 4. This is necessary if pupils are to deduce the fractional part of the whole amount each person receives when separations are made. Without reference to concrete things, pupils find it difficult to understand that in the case of a 2 to 1 division of a number of things the shares are $\frac{2}{3}$ and $\frac{1}{3}$, for 3 to 1, $\frac{3}{4}$ and $\frac{1}{4}$, and for 4 to 1, $\frac{4}{5}$ and $\frac{1}{5}$. Likewise, that if the ratios are 1 to 2, 1 to 3, and 1 to 4, written as the fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, the parts are $\frac{1}{3}$ and $\frac{2}{3}$, $\frac{1}{4}$ and $\frac{3}{4}$, and $\frac{1}{5}$ and $\frac{4}{5}$, respectively. Unless children are given careful guidance in the development of the ability to deal with simple ratios, it is not uncommon for them to indicate that the shares in a 19 to 1 division are $\frac{18}{19}$ and $\frac{1}{19}$ rather than $\frac{19}{20}$ and $\frac{1}{20}$, nor is it likely that this concept can be acquired without the use of learning aids.

Ratios derived from a comparison of two numbers for which the ratio is not an integer are even more difficult to understand, i.e., ratios of 3 to 2, 4 to 3, 5 to 2, 5 to 3, and the like. Understanding may be gained with the aid of objects such as nickels, which have numerical value. It is possible to pose problems such as: What would be the shares if 25¢ (5 nickels) were divided on a 3 to 2 basis; 35¢ (7 nickels) on a 4 to 3; 35¢

(7 nickels) on a 5 to 2, and 40¢ (8 nickels) on a 5 to 3 basis? From solving problems such as these, with concrete things, pupils may be led to make deductions leading to the meanings of ratios expressed in terms of 3 to 2 4 to 3, 5 to 2, and 5 to 3. The further step of stating the ratios as proper fractions may also be taken and problems solved which involve ratios of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{3}{5}$, and the like. Unless a ratio expressed as a fraction, such as $\frac{2}{3}$, is related to its reverse, $\frac{3}{2}$ or 3 to 2, it is difficult for pupils to appreciate the fact that separation of parts of a whole on a $\frac{2}{3}$ ratio would yield two parts, one $\frac{2}{5}$ of the whole and the other $\frac{3}{5}$, the smaller being $\frac{2}{3}$ of the larger. These are the informal experiences with ratios which should provide the basic meanings required to understand their applications to decimal fractions and percentage.

Again attention is called to the fact that in the world outside of the school, people think of ratios in terms of 2 to 1, 3 to 2, 7 to 6, 8 to 5, and 20 to 1. They make wagers, interpret election results, share profits, and do a host of things without knowing that mathematics texts favor the proper-fraction method of expressing ratios. It would seem that arithmetic teachers might relate their instruction with profit to that which is common in our culture. That is the sensible way of helping children gain basic meanings.

Part-Whole Relationships

There is evidence that the custom of postponing all consideration of part-whole relationships until percentage has been studied is being abandoned. Some children learn to recognize and use the three kinds of part-whole relationships when they acquire the ability to apply their knowledge of common fractions to (*a*) finding a part of a quantity; (*b*) finding what part one quantity is of another; and (*c*) finding the whole quantity when the relative size of a part is known. When decimal fractions are studied, this knowledge is extended to include part-whole relationships as they apply to decimal fractions. Children become skilled in choosing the form, common fraction or decimal fraction, which leads most easily to the answer. To children who have had such experiences, only a slight reorganization of learning is required to include the so-called three cases of percentage in the list of part-whole relationships.

The unification of the three-fold interpretation of part-whole relationships illustrates again the task of the teacher who would teach meaningfully. It suggests the importance of meanings in a program of arithmetic instruction. Without meanings there is no core around which number learning can be built.

In a sense the arithmetic of the middle grades culminates in the union of the part-whole relationships of common fractions, decimal fractions,

and percentage. How well upper-grade pupils, high-school boys and girls, and even adults are able to solve a great many problems of everyday life depends upon how well they understand part-whole relationships and can make the computations with them. Which point of view should middle-grade teachers take toward the content of arithmetic —the one that features meaning, or the one which is primarily concerned with skill in computation? The alternatives are not equally desirable.

Summary

In this presentation, the importance of basic meanings in the number-learning of middle-grade children has been pointed out again and again. As arithmetic courses are now outlined in textbooks and courses of study, insufficient emphasis is placed upon basic meanings. In the chain of refinements which children should make in moving from the concrete to the abstract, they proceed to the abstract at a pace which provides an inadequate opportunity to perceive the connection between the two.

Comparisons have been made to indicate the difference between instruction which seeks full meaning and that which can produce only partial meaning. The difference has been described for the teaching of the major topics in the middle-grade course of instruction in arithmetic. One recommendation has been made throughout the presentation, namely, that children first should be able to solve problems in terms of meanings before algorisms are introduced. This recommendation has been made with the assumption that important generalizations, necessary to satisfy future needs, are more apt to be made in simple than in complex number situations.

Learning situations must be organized and directed in such a manner that children make their own deductions and discoveries. Accordingly, teachers must use methods which give children an opportunity to deal, at firsthand, with meanings, and which give teachers an opportunity to guide, direct, and stimulate children to be active participants in the learning process. At the same time teachers must think of the content of arithmetic in terms of the learner. Especially must they be conscious of the levels or stages of meaning with which children react from time to time in moving from basic experiences with things to the level of symbolic thinking.

This presentation has been made with the realization that other approaches to a consideration of middle-grade arithmetic might have been made. It is concerned chiefly with method and basic number-meanings because they seem to be the core of an instructional program over which teachers have a large measure of control.

Finally, this presentation was prepared with the realization that there are few research findings which aid in clarifying the procedures suggested. Precisely what the nature of middle-grade arithmetic would be if discovery became the key word of method, and what results might be achieved if meaning became the principle upon which content were organized, is not known. To place the arithmetic of the middle grades on such a basis is a tremendous task. This is, however, the task which appears to challenge those interested in the teaching of arithmetic at this grade level.

CHAPTER VI

ARITHMETIC IN THE JUNIOR-SENIOR HIGH SCHOOL

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Introduction

Some aspects of arithmetic in the junior-senior high school need reconsideration. This need is called to the attention of those interested in the program of the junior-senior high school, not only by the evidence of mathematical illiteracy exhibited by our armed forces during the last war but also by the performance of the graduates of our public school systems in the business world. In fact, the need for some reorientation is frequently indicated by the performance of the very businessmen—the employers of the high-school product—who complain about the graduates of the public school system.

Criticisms of the high-school program usually center around the lack of computational facility of the graduate. However, there is a more fundamental criticism of the program of the junior-senior high school than the mere failure to get the right total on a bill of sale. The failure to get the right sum for a column of numbers or the failure to find the proper result in a percentage situation is merely a symptom of the real difficulty. The public school program is in difficulty for two very fundamental reasons: (*a*) Arithmetic is still taught as a series of rules which produce the right answer to isolated number situations (provided the student can remember the rules). There is much evidence that arithmetic is not generally taught as the application of a comparatively few mathematical generalizations about quantitative situations, situations which have social significance. Present-day practices still place more emphasis on the final result than on the methods of thinking employed by the pupil. (*b*) Mathematics teachers in general are, as yet, not convinced that education is for all American youth. That is,

> We mean that youth, with their human similarities and their equally human differences, shall have educational services and opportunities suited to their personal needs and sufficient for the successful operation of a free and democratic society.[1]

[1] *Education for All American Youth,* p. 17. Washington: Educational Policies Commission of the National Education Association and the American Association of School Administrators, 1944.

In particular, secondary-mathematics teachers are not, on the whole, convinced that education in the secondary school is for all American youth according to their needs. Teachers of secondary mathematics are being asked to make fundamental changes in their philosophy of education and to add materially to their offerings. Both of these requests involve "growth processes"; both require time.

FACTORS INFLUENCING JUNIOR-SENIOR HIGH SCHOOL ARITHMETIC

There are those interested in the program of the junior-senior high school who grudgingly admit that some knowledge of arithmetic is necessary. They will contend that the time allotted to arithmetic in the elementary-school program could be devoted to more important "educational experiences." The influence of this group waxes and wanes, depending on the particular educational philosophy that happens to be influencing the general climate of opinion in the educational world. The result is that teachers are never quite sure what place arithmetic should have in the curriculum. Objective evidence of the uncertainty can be found in those elementary schools which organize an integrated or a core program without bringing the mathematics of the elementary school into the core.

This situation, in part, results from the fact that mathematics has been isolated too long from other areas of interest in the public school program. The reason for this isolation is not hard to find. If the chief end of arithmetic is to obtain the correct answer to a verbal problem found in an arithmetic text or to an example such as 15 per cent of 142, then one will find considerable difficulty in seeing how mathematics can be used in the social-studies field or even in the field of elementary science. This is the kind of arithmetic with which most of our social-studies and science teachers are acquainted.

On the other hand, if the ultimate goals of arithmetic are to establish patterns of thought, to develop generalizations, and to establish means of attacking real problem situations, then innumerable instances can be found in which arithmetic can contribute to a core program, a social-science program, or a science program. For example, one can teach that the British Isles import most of their wheat and wheat products. It is also possible to teach that the British Isles import 85 per cent of their wheat products. In one case the pupil has a vague idea of the problem Britain faces in getting wheat for its bread, breakfast foods, and wheat products. In the other case the pupil knows that out of every 100 bushels of wheat products used by the British Isles, 85 bushels of wheat must be imported. Similarly, in science it is possible to teach that the sun is lower in the noonday sky during the winter months than it is during the summer

months. However, it is also possible to raise the question as to whether the altitude of the noonday sun varies during the year. As a result of the subsequent discussion the pupils might, happily, decide to solve the problem by collecting pertinent data. This could easily be done by setting up a shadow stick, tabulating the length of the shadow from day to day, and computing the angle of elevation of the sun. A study of the data would show that the altitude of the sun is not fixed.

The distinction between the two procedures, so briefly described above, is a crucial one for arithmetic in the junior and senior high schools. Arithmetic taught as isolated facts will not fuse with the total program of the junior high school; in fact, such procedures will not even fuse the subject of arithmetic with itself. From this point of view it could even be said that, to the extent that arithmetic is taught as isolated facts, it ceases to be arithmetic. On the other hand, the teaching of arithmetic as a means of attacking problems of concern to the welfare of mankind, and as significant patterns of thought, does enable it to fuse, both with itself as arithmetic and with the general program of the junior high school. Fawcett makes the following comment regarding the place of mathematics in a core curriculum.

> An increasing body of opinion supports the position that these highly desirable results can best be achieved through the core curriculum, and the vigorous voices of many students unite in support of this position. Through their school experiences they have learned to define problems, to gather relevant data, to formulate hypotheses, and to test their validity. They know what it means to make use of a wide variety of resources in their study of a given problem, and they recognize the limitations of a single reference.... The problem ... is to consider the relation of mathematics to this kind of a program, and that is a problem which should claim the attention of all thoughtful teachers in this important field of learning.[2]

It is questionable whether all the important mathematical concepts needed by the citizens of a democratic society can be taught in a core curriculum. However, as Fawcett further explains, "The core can be helpful in emphasizing the importance of mathematical concepts and principles, and it is quite possible that core activities can provide a readiness for the study of these ideas."[3] On the other hand, mathematics can be helpful to the core curriculum only if it contributes insight, generalizations, and significant patterns of thought so as to clarify the core meanings.

Failure to make a fundamental change in the ultimate objectives of

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[2] Harold P. Fawcett, "Mathematics and the Core Curriculum," *Mathematics Teacher*, XLII (January, 1949), 7.

[3] *Ibid.*, p. 11.

arithmetic has bolstered the case of those who see little or no reason for keeping it in the curriculum as a field of study or for incorporating it in an integrated program. As a result, the time devoted to arithmetic has, in some instances, been drastically cut down in the first two years of the junior high school. Moreover, in some of these schools little or no effort is made to bring the pupils up to a minimum level of ability to think in quantitative terms. There is a good reason to question whether the best interests of the pupils, and especially of the lower-ability groups, are served by such practices.

On the other hand, the practice of requiring a course in arithmetic in the last year of the junior high school for those pupils in the lower-ability group is at times a rather short-sighted policy if the methodology and the content of such a course are not carefully considered. When a student who has already acquired a set of psychological blocks against learning some of the abstract generalizations of arithmetic is asked to take another course, taught largely by the same methods and having the same content, he is doomed to failure in a large majority of the cases. This type of student has had trouble grasping the generalizations of arithmetic for the past seven or eight years. In such cases, the student, and sometimes the teacher as well, is convinced that "another year's try" will not be profitable. However, since the pupil is caught in the mesh, he submits with good grace and sits through another semester, or another year, of paper-and-pencil work in arithmetic.

The pupils in the lower-ability group should be freed from incessant drill and routine work. It would be to their advantage to be placed in a laboratory situation, possibly the science laboratory, which requires some quantitative thinking and some subsequent computation. In the laboratory the pupil can usually be placed in a more realistic and interesting situation. Furthermore, a greater variety of motivating techniques are available in a laboratory situation than are available in the usual mathematics classroom equipped only with paper, pencils, and workbooks.

Basic Concepts Pertaining to Junior-Senior High School Arithmetic

In a systematic study of the junior-senior high school arithmetic program it is always advantageous to look for fundamental principles which will guide the thought processes toward assembling an integrated body of subject matter. Such a group of fundamental principles is not only essential for developing a curriculum in arithmetic, it is essential for the pupil if he is to assimilate anything other than isolated facts from the sequences of experiences which constitute the curriculum. In other words, the basic principles used by the teacher to develop an integrated mathe-

matics course are used by the pupil to integrate mathematical understandings.

Some General Considerations

In the literature on the subject of arithmetic there is much discussion about the "social phase" of arithmetic and the "mathematical phase" of arithmetic. The "social phase" is generally interpreted to mean the uses of arithmetic in daily living. The "mathematical phase" of arithmetic is interpreted to mean the study of the interrelationships of numbers, number combinations, and number symbols. This dichotomizing of the objectives of arithmetic tends to divide the teachers of arithmetic, as well. There are those who say that only the "useful" parts of arithmetic should be taught, while there are others who contend that arithmetic, being a logical subject, must include some techniques which are not "useful" but which help to attain the essential logical unity, thereby assuring a more meaningful arithmetic. For this reason, one will find widely divergent practices in arithmetic classes because of different degrees of emphasis on one or the other of the two "phases" of arithmetic instruction. This condition would seem to indicate that teachers, as a group, are not very sure whether arithmetic has only one basic objective or a dual objective.

It would seem that there is a broad field of common interest that is ignored by proponents of both the "mathematical phase" and the "social phase." Arithmetic is essentially a language, used to express relationships observed in many phenomena. In its most significant aspects, this language is put to use in resolving a problem situation. Arithmetic may be applied to phenomena of the kind so ardently approved as a subject of study by the "social-phase" proponents of arithmetic: for example, the buying of, and paying for, eleven gallons of gasoline. Many of the acts involved in paying taxes can be described by the symbolism of arithmetic.

On the other hand, the phenomena to which arithmetic is applied may be of an entirely different nature. Arithmetic can be used to express the relationship between distance, rate, and time; it can be used to describe the variations in temperature that occur during any period of time; it can be used in the process of collecting and interpreting the data obtained in almost any problem situation whether of an economic, social, scientific, or mathematical nature.

Thus arithmetic is used, at times, as a means of communicating quantitative ideas and the relationships between quantities. In this role, arithmetic may be said to have a "descriptive function." Then again, arithmetic is used as a powerful instrument for solving certain kinds of problem situations. In the latter case the communications aspect of arithmetic may be very clearly present or it may be entirely absent, as in those instances in which a single individual is trying to solve a problem.

Teachers of arithmetic should consider whether it is not more fruitful to think of arithmetic in terms of its description of characteristics of physical phenomena and whether it is not the duty of the arithmetic teacher to show how arithmetic does describe quantitative situations of all kinds—social, economic, scientific, and mathematical. In fact, there are those who would contend that the basic objective of arithmetic is to teach the high-school pupil that arithmetic is valuable because it is the language that can be used to describe so many different quantitative situations. Furthermore, these same individuals would contend that it is the duty of the teacher of arithmetic to so select learning experiences for the arithmetic class that the pupils will see the advantage of using arithmetic for describing quantitative relations in a great variety of situations. Selecting a variety of experiences which emphasize the need for arithmetic as a language would necessarily include some experiences in which the "social phase" was predominant as well as experiences in which the "mathematical phase" was predominant. Without such a variety, the pupil would not realize the power of the language. Without a variety of experiences, the pupil would not come to know the language of arithmetic thoroughly—how it is used, when it is used, and why it is used.

In any study of the junior high school program the interplay of the "social phase," the "mathematical phase," and the "descriptive function" of arithmetic should be given serious thought. The type of program developed will depend upon the degree to which their interrelationships are understood and the degree to which the staff emphasizes one or another of these three ideas.

Teachers who decide to accept the descriptive function of mathematic as the central theme of the program of junior high school mathematics will be confronted with two kinds of implementive problems: (*a*) determining what understandings, meanings, generalizations, and skills are necessary to use arithmetic as a descriptive instrument, and (*b*) selecting a variety of experiences which will develop the necessary facility in using this instrument to describe physical phenomena. It will be the purpose of the next few sections to enlarge on these two general problems.

Understandings Essential for the Ready Use of the Language of Mathematics[4]

In order to unify the concepts taught in junior high school arithmetic, it is essential that the teacher keep in mind certain central mathematical

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[4] Many of the ideas in this section were gleaned from *Mathematics in General Education: A Report of the Committee on the Function of Mathematics in General Education* (New York: D. Appleton-Century Co., Inc., 1940); and *The Place of Mathe-*

ideas. Without these central ideas the work in arithmetic is likely to degenerate into the teaching of a series of topics which are to some extent unrelated. Such procedures fail to provide the child with central ideas around which he can organize his knowledge of arithmetic. Furthermore, central ideas provide the teacher with a few fundamental goals which can be kept in mind throughout the year. Keeping these goals in mind will produce better results than will be possible if the teacher presents a series of topics without fundamental unifying themes. Central themes can unify the entire instructional program in the junior high school if they are maintained throughout the three years.[6] It will be the purpose of this section to list and briefly discuss a few of these fundamental ideas.

Measurement. Measurement consists in selecting a unit possessing physical characteristics similar to the quantity to be measured and comparing this unit with the thing to be measured. For example, it is possible to give the dimensions of a desk as 6 pencils long and 3 pencils wide. In this case the pencil is the unit of length, and this unit is compared with the distance between two opposite edges of the desk in order to arrive at a numerical figure which is descriptive of the length of the desk.

Similarly, when measuring an area it is possible, and desirable in initial teaching experiences, to use a square other than the unit square in order to find how many of these squares would be required to cover the top of a given desk. It would then be possible to say that 160 such squares give a measure of the amount of desk-top surface. Once this concept is grasped by the junior high school pupil it is then easy to show the need for a standard unit of measure, such as the inch and square inch.

The fact that all measurement is a process of approximation is another concept fundamental for an adequate understanding of the meaning of measurement. It is particularly fundamental if one is to compute, intelligently, with approximate numbers. The failure to teach the intelligent use of approximate numbers is probably one of the more obvious failures in junior-senior high school arithmetic. Pupils should understand that adding two numbers such as 15.6 inches and 34.567 inches, without rounding both to an appropriate degree of accuracy, involves some useless operations. The rounding of these numbers to the nearest tenth of an inch before adding should be as automatic as the response to the addition combinations needed for finding the sum. Similarly, the pupils should be lead toward an understanding of the futility of using 3.1416 for

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matics in Secondary Education, chap. iv (Fifteenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1940).

[6] For a detailed illustration see H. C. Trimble, "An Idea," *Mathematics Teacher*, XLI (January, 1948), 36–39.

pi (π) when finding the circumference of a circle whose diameter is 16.4 inches.

Measured numbers occur so frequently in junior high school studies and in daily life that the intelligent use of these numbers is an important objective.[6]

Comparison of Numbers. Comparison of numbers is one of the more frequently used general ideas of arithmetic. Such common phrases as "how much larger" and "how many times" suggest the use of comparison techniques. Furthermore, all of ratio, percentage, and index numbers, some of the work in common fractions, and parts of graphing are simply different techniques used to compare numbers.

Teachers interested in integrating the instructional program in arithmetic in the junior and senior high school should note how this idea threads throughout the curriculum of the elementary school, the junior high school, and the senior high school. Thus, in the second or third grade the pupil may spell correctly 18 out of the 20 words assigned for the week. The symbol $\frac{18}{20}$ correctly expresses his performance for the week in spelling. This symbol has meaning for the child. Here is an initial experience requiring that two numbers be compared. In adult terminology this is a ratio. When the child studies subtraction he again meets the comparison idea. John has 6 apples and Mary has 4 apples. How many more apples does John have than Mary? The child compares the number of apples John has with the number of apples Mary has.

In the study of fractions, the teacher should not lose sight of the fact that $\frac{3}{4}$ is a ratio as well as a part of a whole or group. When studying decimal fractions, the pupil learns that these ratios can be expressed in different ways. Then finally, for the junior high school, the pupil learns that percentage is only another way of talking about ratios; that is, a record of $\frac{18}{20}$ for the week may also be expressed as $\frac{90}{100}$. So for the seventh-grade pupil who is starting his study of percentage, 90% means 90 out of 100. This approach merely relates a new symbol (%) to an old idea.

Operations. The concept of operation in arithmetic is traditionally a neglected one. This is probably a result of the dominance of the connectionist theory of learning in the teaching of arithmetic during the period from 1920 to about 1940. If learning the sum of 6 and 9 is merely a memory process from the initial experience to the final mature response—in the connectionist theory of learning, these two responses should be identical—then the role played by the symbol, +, is of minor importance. From the very beginning of the child's arithmetical learnings, connectionism

[6] For an extended treatment of this topic see Aaron Bakst, *Approximate Computation.* Twelfth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1937.

pushes the concept of operations into the background. The symbols of operation serve only as stimuli which elicit different responses when two numbers connected by a $+$ or $-$ are presented. In such a situation it would be an accident if the child grasps the fundamental role played by the operational symbols in arithmetic.

In order to clarify the point under discussion, one or two illustrations will be taken from the work in the elementary school. Assume that the child knows the meaning of the symbols 9 and 8, as well as an understanding of the number system. What should be the meaning of $9 + 8$? The plus sign indicates that the groups of 9 and 8 are to be organized into groups of tens and ones, that is, 1 ten and 7 ones. Thus the symbol, $+$, plays a vital role. It indicates the operation to be performed—a real physical act—that of regrouping by tens. Similarly, the symbol, $-$, in the exercise $9 - 3$. The $-$ indicates that 3 objects are to be separated from the 9 objects. Thus the $-$ indicates a physical operation (the taking away process) that is to be performed.

An exercise that is often encountered in the junior high school is written as $\frac{2}{3}$ of 6, or $\frac{2}{3} \times 6$. Here the sign, $\times$, means that the 6 objects are to be divided into three equal groups and that two of these groups are to be set aside for special consideration. Here, again, the $\times$ has an operational meaning—an action that can be physically performed in the initial learning experiences.

In percentage, the operational process involved in taking 16 per cent of 200 can be physically performed, or visualized, by thinking of 16 per cent as being 16 out of every 100, or 16 per 100. Then 16 out of every 100 results in taking 32 out of 200. Here, again, the physical operation is highlighted in order to establish the meaning of the $\times$ in the initial learning experiences in percentage.

Once the concept of performing these operations has been established, the question arises as to how numbers "behave" when subjected to operations such as addition, subtraction, multiplication, division, raising to a power, and extracting roots. Furthermore, it is natural to ask, do these operations have similar or the same properties? For example, does $16 + 9 = 9 + 16$? Does $13 - 5 = 5 - 13$? Does $\sqrt{9 + 16} = \sqrt{9} + \sqrt{16}$? Is the ratio of $\frac{3}{4}$ the same as the ratio of $\frac{4}{3}$? The reader can supply hundreds of similar questions which bring out an important property of the operations studied in junior high school mathematics. If the pupil understands the operation, the answers to such questions are immediately obvious. Without focusing the pupil's attention on the role played by the operations, there is some question as to whether a well-rounded concept or an integrated concept of arithmetic will develop.

Fundamental Skills. Developing a certain degree of facility in comput-

ing is, of course, essential in any arithmetic program in the junior and senior high school. However, to let this objective consume too much of the time allocated to arithmetic would unbalance the program. Furthermore, too great attention to developing computational facility will result in neglecting some other feature of the program which may be just as essential as computational skill. Great care must be exercised to prevent computational skills from blocking progress toward attaining a unified arithmetic. Skills cannot integrate arithmetical learnings. However, they can, if properly used, aid in attaining these learnings. The pupil with the necessary skills is free to develop his thinking on a higher level than he would be if he had to concentrate on the mechanics of an exercise every time he encountered it. Skill frees the pupil to reorganize his learnings, frees him to reach broader generalizations, and suggests new procedures and new processes. However, these higher levels of learning are not achieved by skill alone. The generalizations outlined in this section are essential, and they more closely approximate the ultimate goal in arithmetic than does the computational objective. In its proper place computational skill enables the pupil to reach higher levels of power, but instructional techniques and goals must be so adjusted as to carry the pupil to these higher levels of power. The skill level is not the ultimate goal.

No more need be said about the place of computational skill in the arithmetic program of the junior high school. The subject has been well treated in many standard references. In particular, the reader is referred to the "Second Report of the Commission on Postwar Plans."[7] While the report does not deal with skills alone and certainly emphasizes the tragedy of teaching arithmetic as a skills course, it states a definition of functional competence which deserves considerable thought.

Dependence. Much attention is given in the study of mathematics to how quantities depend upon one another or how quantities "vary together." For example, the circumference of a circle depends on the length of its diameter; or the distance that a car will travel in a given period of time depends upon its rate of travel. In a much less direct way the quality and quantity of food a given family will eat depends upon the annual income of the family.

Much of our daily thinking is also concerned with how one set of numbers depends upon another set of numbers or how certain conditions, which are more or less quantitative, depend on other sets of conditions. For this reason, and because it is an essential element in problem-solving, the ability to think in terms of how numbers are related is important. It

[7] "The Second Report of the Commission on Postwar Plans," *Mathematics Teacher*, XXXVIII (May, 1945), 195–220.

is also important to be able to express this relationship through such mathematical techniques as the graph, the formula, the worded statement, and a table of numbers.

Problem-solving. There are those interested in the teaching of arithmetic who contend that problem-solving should be the ultimate objective of any course in mathematics in the junior and senior high school. From this viewpoint such concepts as operations, comparison, dependence, and fundamental skills should be subordinated to problem-solving because all these concepts are needed in problem-solving situations. However, it is easier to keep in mind that problem-solving ability is a complex ability and, as such, requires a knowledge of various general techniques. The instructional program can then be planned so that problems are solved through the use of various techniques or combinations of techniques.

The term, problem-solving, is used in this chapter to mean that activity, or those activities, engaged in when a class or an individual faces a puzzling situation which is to be resolved. In the process of solving the problem the individual may engage in a number of activities all more or less related to finding a solution. First of all he may wish to determine what is causing the puzzling situation. In other words, he formulates his problem. After his problem has been formulated he may make a guess as to what the answer ought to be, or he may not be ready to formulate a plausible guess. In any case he would undoubtedly gather some pertinent information bearing upon the problem in question, with the hope that he might verify his "guessed solution" or show a need for more data. In either case he will collect data, organize data, interpret data, compare numbers, measure, and observe relationships between quantities in order to arrive at new conclusions which might explain, verify, or reinforce, his hypothesis—his guess.

Problem-solving, from this viewpoint, is a complex process and does not involve a mere decision as to which of two or three possible operations are to be performed in order to obtain a given numerical answer. So conceived, problem-solving is socially more significant than is the present-day interpretation of this term. In its broadest sense, problem-solving teaches attitudes of considerable value to a democratic society. It teaches the art of suspending judgment, carefully weighing facts, making value judgments, and arriving at conclusions on the basis of a preponderance of evidence. It may even lead to a decision when most of the evidence points toward accepting a certain solution as merely the most plausible solution.

"But," the teacher will say, "these problems are not in the textbook." Of course not. Many or most of the really good problems cannot be found in textbooks. Do teachers have time for all the work required to initiate

such a program? Many teachers have the time.[8] If each semester a class were to attack a problem that was a significant problem for that class, and attack only one problem, then over a term of years arithmetic teachers would find that they were in possession of a collection of problem situations which would prove to be of interest to both students and teacher. The time required, on the part of the teacher, would not be excessive if there were a gradual transition from a traditional program to one which emphasizes problem-solving in its broader aspects.

This brings to an end the discussion of a few fundamental ideas generally used to unify the arithmetic program. It is not a long list. It can easily be kept in mind by any teacher. Furthermore, these ideas afford a very convenient means of unifying a course in arithmetic. For example, if a problem involves the comparison of two numbers, the teacher can raise the question, "How do we compare numbers?" "Is this the method which best expresses the comparison we wish to make?" If the problem at hand necessitates expressing a relationship between two or more quantities, or sets of quantities, the question naturally arises, "Which of the several methods known to us is best suited for our purposes?" "Why is a table or a graph a better method, in this case, than is the verbal statement?" Such questions tend to unify the various aspects of arithmetic by contrasting purposes, techniques, form, and usefulness. Both the pupil and the teacher can focus attention on important general ideas, disregarding incidentals. The teacher can review her program, look at her text, and say, "If I succeed in teaching these basic ideas it really doesn't make much difference whether we do the work found on pages 156–70 in the text. My class has already had a number of experiences with the ideas taken up on these pages."

Selecting Classroom Experiences

Some teachers will be sorely taxed to find good problem-solving situations, particularly if problem-solving in the broadest sense of the word has been neglected. The discussion has already referred to sources listing problem-solving situations. The resourceful teacher will be able to find many others. In fact, a teacher must be resourceful enough to find good instructional material if the pupil is to feel that he is working on something of value to himself. It is highly improbable that all the necessary data and sources of information can be placed in textbook form or organized in such a way that a busy teacher can quickly assemble it and say, "Now I have everything I'll need. Tomorrow I can introduce this new unit in my ninth-grade mathematics class." Such a procedure would

[8] See, for example, Harriette Burr, "Mathematics in General Education," *Mathematics Teacher*, XL (February, 1947), 58–61.

immediately take the life out of the problem. The element of doubt, the searching for information, and at times not knowing where to turn, are all important elements of problem-solving situations.

Space will permit the listing of only a few areas in which problems of value and interest to a junior high-school class might arise. The teacher can readily identify a number of good sources from which additional ideas can be obtained.

a) *Social-economic Sources:*
1) Local problems in taxation.
2) Old-age pensions—Are they necessary? How many families in the U.S. earn less than $2,000 per annum?
3) Health insurance—Can people pay their own doctor bills?
4) Minority problems.
5) Housing problems.
6) Conservation of natural resources—Are our resources sufficient for future generations at present rate of consumption?
7) Traffic surveys at school crossings.
8) Instalment buying.
9) Budgeting.

b) *Problems from Science:*
1) The relation between the elevation of the sun and the seasons of the year.
2) What conditions influence the dehydration of vegetables?
3) How would one fly from New York to Mexico City?
4) The relationship between soil erosion, the slope of the hill, type of soil, and kind of coverage.
5) The principle of the lever (a laboratory problem).
6) Determining the mechanical advantages of pulleys.
7) Testing diets.

A brief bibliography has been included at the end of this chapter. The pamphlets, magazine articles, and books listed in this bibliography have additional suggestions for the teacher interested in problem-solving.

The Senior High School Program

There is considerable evidence of a growing awareness of the need for a program in mathematics in the senior high school which emphasizes some of the basic mathematical ideas mentioned in the sections on junior high school mathematics. The reasons for this increased interest are numerous. To list only a few: (*a*) Many high-school graduates are woefully weak in their ability to think quantitatively. (*b*) Many high-school graduates are also weak in computational skills. (*c*) Teachers are beginning to doubt the traditional grade placement of many of the topics in the junior high school, as, for example, instalment buying, taxation, insurance, stocks, and bonds. Students in the junior high school are not, as yet, ma-

ture enough to find these topics of interest. Furthermore, many of the students in the senior high school will need some knowledge of these socially important topics after graduation if, indeed, they are not in need of them, or using them, in their senior high school days. (*d*) A growing awareness on the part of teachers of mathematics that the traditional program in mathematics cannot satisfy the needs of the entire high-school population. The Commission on Postwar Plans of the National Council of Teachers of Mathematics emphasized the need for such a program in its second report. Thesis 19 of this report states, "New and better courses should be provided in the high schools for a large fraction of the school's population whose mathematical needs are not well met in the traditional sequential courses."[9]

There are a number of conditions which tend to retard any movement toward the broadening of the mathematics program in the high school. In the first place there is some lack of material, particularly if an extensive program of mathematics for all the students in the senior high school is contemplated. However, there are excellent materials on the market for a one- or two-semester course in eleventh- or twelfth-year mathematics which emphasizes the needs of the consumer. In the second place, mathematics teachers are reluctant to devote a part of their time to a program in mathematics designed to aid all of the students in the high school. The courses in algebra, geometry, and trigonometry prove to be more attractive courses because most teachers were trained in that kind of subject matter and because it takes less work to teach conventional courses which have been taught for a number of years. Third, there is some doubt in the minds of a large majority of the mathematics teachers whether such topics as insurance, taxation, budgeting, and general consumer problems belong in the field of mathematics. In fact, many mathematics teachers object to taking time in a mathematics class to collect data, formulate problems, or discuss social implications of data collected. Such activities, they say, are not in the field of mathematics and, hence, should not consume the time of a mathematics class.

On the other hand, there is an ever increasing number of teachers in the field of mathematics who are struggling with the problem of developing a senior high school program in mathematics for every student according to his needs. Certain California schools, for example, are doing an excellent piece of work based on the philosophy that the mathematics teachers are responsible for completing the mathematical education of the slow learner in the senior high school and that the program in mathe-

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[9] "The Second Report of the Commission on Postwar Plans," *op. cit.*, p. 210.

matics should be such that, in so far as is possible, it meets the needs of the slow learner as well as the needs of the fast learner.[10] While much of this program is still quite new, it seems to have progressed far enough to serve notice on the educational world that it is worth watching.

A good case can be made for a series of courses in the large senior high school which are oriented to serve the needs of students with different mathematical backgrounds. There is need to consider courses designed for the student with a very meager mathematical background—courses designed to develop an understanding of the number system and the ability to think in terms of number—and courses designed to erase some of the psychological blocks against learning arithmetic which the pupil may have picked up in his previous school experience with the subject. Other high-school courses should be offered for the student who needs a "brush up" on junior high school arithmetic before entering specialized courses in industrial mathematics or the regular courses in algebra and geometry. Still other courses are needed for the senior year to present problems connected with such topics as insurance, instalment buying, taxation, and investments—courses which may serve as the student's last contact with mathematics in the senior high school.

What name would one give to these courses? They need no name other than Mathematics I, Mathematics II . . Mathematics V. In fact, there is reason to believe that the traditional courses in algebra and geometry might well find their place in the sequence of courses designated by number alone. This would help prevent stigmatizing certain courses as "dumb-bell courses." Would this practice help break down the boundaries between algebra and geometry?

Needless to say, any such extensive offerings in mathematics should be accompanied by a good counselling program. Data on individual students must be available to the counsellor who is to help them select the proper mathematics courses. Without a counselling program, such extensive offerings in mathematics could not function efficiently in serving the best interests of the pupils. A pupil who is placed in a class above or below his own mathematical background will not be in a healthy learning atmosphere. Furthermore, each student should be informed as to the needs —the mathematical needs—in the vocation of his probable choice.[11]

The smaller high schools will be confronted with a different problem.

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[10] Dale Carpenter, "Planning a Secondary-Mathematics Curriculum To Meet the Needs of All Students," *Mathematics Teacher*, XLII (January, 1949), 41–48.

[11] See *Guidance Pamphlet in Mathematics for High-School Students.* Final Report of the Commission on Postwar Plans of the National Council of Teachers of Mathematics. New York: *Mathematics Teacher*, 1947.

They have neither the staff nor the large enrolment needed to justify offering such a diversified program, that is, a program in which separate classes are organized to remedy, or develop, specific kinds of arithmetical concepts. However, the teachers in these schools can develop laboratory techniques and organize classes in such a way that small groups of students with different arithmetical backgrounds can be taught during the same class period. The student in such courses could be given credit for the course each time he took the course and worked with a different group simply by reporting his credit as Mathematics I or Mathematics II.

There is a need for a more diversified mathematics program in the senior high school. It stands as a challenge to the teachers of mathematics and the administrators of our public school systems to arrive at a program which will meet these needs. Those in the educational world who are in touch with some of the undercurrents in educational movements sense the coming of such a program. In fact, as has been pointed out, some school systems have such a program in operation. The traditional courses in algebra and geometry will, of course, always be in the total program, but they will be only a part of the program—the part designed for those of special interests and special abilities. The majority of high-school pupils will be registered in courses which come to grips with problems of greater immediate social significance. These courses should seek to develop sound mathematical concepts. Furthermore, these courses should use these mathematical concepts in such a way that the pupil will realize that mathematics enables one to live a richer life—to better understand the world in which he lives.

Senior high school courses, like junior high school courses, should not become drill courses in arithmetic. Workbook exercises, and page after page of drill, will go far to destroy that breath of life which makes the difference between a good arithmetic course and a poor one. In developing the senior high school program it would be well to keep in mind that "number manipulation in its proper position is minor in importance to the development of sound conceptual patterns and the mastery of procedures for clear thinking and accurate expression. Number bears about the same relationship to mathematics that the alphabet bears to vernacular language, or that the musical notation system bears to musical experience."[12]

Those teachers of mathematics who are doubtful about broadening the usual mathematical program along the lines suggested might well remember that "each of them [the senior high school student] is a human being,

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[12] *Some Mathematics Practices in California Secondary Schools*, p. 4. Bulletin of the State Department of Education, Vol. X, No. 5, June, 1941. Sacramento, California: State Department of Education, 1941.

more precious than material goods or systems of philosophy. Not one of them is to be carelessly wasted. All of them are to be given equal opportunities to live and learn."[13]

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CHAPTER VII

LEARNING AND TEACHING ARITHMETIC

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COMMON INITIAL APPROACHES TO INSTRUCTION

The teaching of arithmetic today shows extreme variation. Markedly different procedures are employed even in classrooms that are considered superior, to say nothing of differences in methods between classrooms representing superior and inferior instruction. Attention will first be given to procedures that are generally considered to be good, those most popular in school systems rated as being good or superior.

The most striking characteristic of the instructional procedures employed in such classrooms is the extensive use of questioning, telling, and explaining in the presentation of facts and processes and in the development of generalizations. In the lower grades the most common method of introducing material is to consider situations (usually from the textbook) which illustrate material to be learned. The questions in the text or some asked by the teacher are considered and either answered by a member of the class or by the teacher. This presentation of a problem situation, or the simpler teacher-question situation, is a very important characteristic of the teaching of arithmetic. For an illustration of the role of questions see Sample Lesson 1 (p. 122). After the initial presentation of a problem or situation, textbook or teacher solutions are offered and explained, and the truth of various statements or aspects of the numbers involved are demonstrated, frequently by child participation. Some more resourceful teachers use projects such as play stores or actual enterprises involving the use of number facts to introduce them, but in the teaching of the fact, the questioning, telling, showing, and explaining is similar to that described above. Reference to use of numbers in the project is similar to the reference made in textbooks to the initial problem.

Following the initial presentation of a fact or process, children engage in various exercises designed to give experience with the fact and with related facts and processes. Usually such exercises involve the use of simple problems similar to the original illustrative problem of text or project. As a part of these exercises, objects or drawings are often used to show that

computations have been correctly performed. While children are encouraged to use the best methods or procedures, immature and less proficient means are recognized and even considered by some teachers to contribute to better understanding. Frequently as a part of the experience exercises, some rationalization is attempted. An example is the development of a rule for finding the sum of two consecutive numbers. This is done through experience exercises showing that $4 + 5 = 4 + 4 + 1$, $3 + 2 = 2 + 2 + 1$, and so on, followed by questions designed to bring out the relationships that exist, e.g., "How does the sum of 4 and 5 compare with the sum of 4 and 4? 5 and 5?"

When the teacher is satisfied that the majority of the class understands the new fact or process, practice exercises for mastery are introduced. For an illustration of some of the instructional procedures mentioned, see Sample Lesson 2.

In upper-grade arithmetic less emphasis is placed on development and more is given to explanation. In some instances problems or situations to illustrate new phases of arithmetic being taught are dispensed with and the first work consists of explanations and demonstrations. There is also less reliance on questioning and exercises which provide experiences. Problems and practice exercises are probably more prevalent, as are tests and the use of the blackboard. For an illustration of some of the points listed, see Sample Lesson 3.

In the upper-grade classrooms where the instructional procedures described are used, there is no particular concern over variations in achievement among pupils. In fact, such differences are recognized as being normal, and attempts are made to adjust questions, suggestions, and work assignments to the ability of individuals. No one predetermined level of performance is expected.

In large numbers of classrooms instructional procedures are used which in emphasis and method differ somewhat from those described in the preceding paragraphs. While not inferior, and certainly not inferior in the sense that they characterize the work of incompetent teachers, these procedures are not, as has already been stated, the most widely used. For want of better terms, these two types of procedures will be referred to as type 1 and type 2.

One of the most obvious differences between type 1 and type 2 procedures is found in the general attitude toward the subject matter of arithmetic. In the type 2 classrooms all attention is centered on presentation of the facts and processes in textbook or course of study, without the developmental steps involving questions, demonstrations,explanations, and showing that characterize type 1 instructional procedures. In classrooms where instruction is of type 2, subject matter is usually divided into small

portions for presentation. Explanations are brief with practically no discussion or participation by pupils. In fact, the textbook explanation, which because of the very nature of books has to be brief and in language which is not always entirely familiar to the children, is frequently the only explanation offered. The arithmetic period is then primarily a work period for solving examples and problems. Ability to secure correct answers without much outward concern for understanding is the rule. It should be noted that most teachers using such procedures believe that understanding is concomitant with ability to compute and to solve verbal problems and other exercises. Generalizations, such as the sum of two consecutive numbers (e.g., $5 + 4$) is 1 greater than the double of the smaller number, are stated rather than developed through consideration of questions and exercises. It is common to find that only one procedure or solution is acceptable, and it is generally assumed (although not actually true) that all pupils are to master that which is presented. Sample Lesson 4 illustrates some of the type 2 procedures.

Sample Lesson 1. (This sample is only part of a lesson based on material in a third-grade text. It illustrates the use of textbook situations and the important role of questions in instructional procedures.)

This lesson began with the teacher saying, "While Mary and Bob pass out our books, let's see how many uses of number we can find on the board. Who wants to give the first one?"

The date, page references in reading assignments on the board, cost of milk per week in a reminder notice, and the like, were some of the uses of number that were cited. When Mary and Bob had finished their task, the teacher said,

"Now let's open our books to page 43. (Pause) How many people are seated at the table in the picture?" (A picture showing 5 children at a table occupied the top one-third of page 43.)

After the answer to the last question and some interesting comments about the picture, a pupil voluntarily read this passage from the book: "Miss Hill asked Alice to get one piece of colored paper for each child in the class. Alice wondered how many pieces of paper that would be."

"How could she find out?" was a question inserted by the teacher at this point. The answer was given immediately: "To find out, she must first count the children. Then she must get as many pieces of paper as there are children."

The reading continued through the following four exercises in the book.

1. Five children sit in each group. There are 6 groups. Can you find the number of children by counting by 5's?

After this book question, several children answered. One said, "Sure you can. You could say 5, 10, 15, 20, and so on."

"Are you sure you would have the right amount when you count by 5's?" asked the teacher. "Look at the 6 groups of dots in your book."

"How many dots in each group?" (Pause for pupils to give answer.) "Could we use these dots to see if counting by 5's would give the number of pieces of paper Alice needs?"

Several minutes were spent in considering the answers, comments, and suggestions concerning the last question and the problem it suggested.

2. Alice said there are 30 children. Is she right?

3. The colored paper is in 10-sheet packages. If you had been Alice, how many packages would you have picked up?

She picked up one package and said, "10." She picked up another and said, "20," and a third and said, "30."

4. Alice started giving sheets of paper to the children. Did she have one for every child?

Attention is called to the fact that even though the textbook presentation of this material included four questions, the teacher used five more questions in directing the lesson up to exercise 2. Notice, too, that the textbook question in exercise 1 is answered in exercise 2. This is in keeping with the accepted practice of posing a problem: If the child can't solve the problem, then solve it for him.

Sample Lesson 2. (Part of a lesson in a fifth-grade class which illustrates the role of text and teacher in the development of a process and shows the use of various ways of getting an answer to a number question.)

"Now turn to page 129 for some new work," was the introduction by the teacher to this part of the lesson. "When you find that page, see if you can tell what the new material is."

After the material had been identified by the page heading, the teacher said, "Let's study together. Look at exercise number 1 and tell what numbers you would add to get the answer to the question."

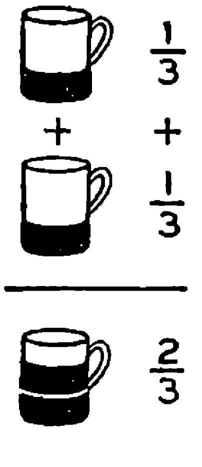
Adding Like Fractions

1. Mother used $\frac{1}{3}$ of a cup of cream in making a cake and another $\frac{1}{3}$ of a cup for making whipped cream. How much cream did she use?

To find the answer you think: $\frac{1}{3}$ cup + $\frac{1}{3}$ cup = ?

One-third and one-third are two-thirds, just as one inch and one inch are two inches.

Mother used $\frac{2}{3}$ cup of cream.



In the consideration of exercise 1 and the explanation, these questions arose. "What does the first drawing show? The second? The one below the line? What is one penny plus one penny?"

In preparation for the definition that followed exercise 1, the teacher said, "We are studying the addition of like fractions. How are one-third and one-third alike?"

The rest of the text material on page 129 is reproduced below:

Fractions such as $\frac{1}{3}$ and $\frac{1}{3}$ and $\frac{1}{5}$ and $\frac{2}{5}$ are called like fractions because they have the same denominator.

2. Name two other pairs of like fractions?

3. When we add inches the sum is inches. When we add thirds the sum is thirds. What is the sum when fourths are added?

4. One-fourth and two-fourths are how many fourths?

5. Study the examples and show that the sums are correct.

a) $\begin{array}{r} \frac{1}{5} \\ +\frac{2}{5} \\ \hline \frac{3}{5} \end{array}$ *b)* $\begin{array}{r} \frac{1}{4} \\ +\frac{1}{4} \\ \hline \frac{2}{4} \end{array}$

c) $\frac{1}{8} + \frac{4}{8} = \frac{5}{8}$

d) $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$

e) 1 pencil + 2 pencils = 3 pencils

f) 1 fourth + 2 fourths = 3 fourths

To add two or more like fractions add their numerators and write the sum over the denominator.

To supplement exercise 2 the teacher asked several other pupils, designated by name, to give a pair of like fractions. "Are $\frac{1}{10}$ and $\frac{7}{10}$ like fractions?" was also asked. Exercise 3 was also supplemented by the question, "What is the sum when eighths are added?"

In their consideration of exercise 5 these questions and suggestions were used: "How was the 3 in the sum for example (*a*) obtained? The 5 in (*c*)? Name the denominators in (*b*). The numerators. Show that the numerators are added in (*d*); in (*f*)."

Twenty minutes were used in the lesson to this point.

In this lesson, as in lesson 1, the teacher used many questions in directing the work of the class. Attention is called to the minuteness with which various parts of the process are developed—suggestion to identify numbers to be added, picturing of the amounts used, simultaneous identification with fractions of the amounts, suggesting computation by use of statement, "$\frac{1}{3}$ cup + $\frac{1}{3}$ cup = ?", calling attention to analogy between fractions and denominate numbers, and so on.

Sample Lesson 3. (A lesson in Grade VII. The topic was initial presentation of case II percentage. It shows the reliance on solutions as explanations and the greater emphasis on practice exercises in instruction than is the case in the lower grades.)

This lesson was begun with the teacher's oral presentation of this situation:

"In the seventh-and-eighth-grade basketball game, Bill made 7 of the

19 shots he tried and Henry made 3 of the 8 shots he tried. Which boy made the greater percentage of his shots?" The teacher wrote the following on the board:

Bill shot 19 times, made 7.
Henry shot 8 times, made 3.

After some discussion by the class, the conclusion was reached that in Bill's case you had to find what per cent 7 is of 19, and, in Henry's case, what per cent 3 is of 8. The teacher then said, "This is the way to find what per cent 7 is of 19. First write it as a fraction, $\frac{7}{19}$. Change that to a decimal, and then change the decimal to a per cent."

```
      .36 = 36%
  19)7.00
     5 7
     1 30
     1 14
       16
```

"Now you try finding the per cent that 3 is of 8."

The first two students to finish were asked to put their work on the board. Attention of the class was called to their work by these statements of the teacher: "Notice how the statement was first written as the fraction, $\frac{3}{8}$. Here the fraction is changed to a decimal, and then the quotient is changed to per cent."

The teacher then directed the attention of the class to their textbook with this statement: "Finding what per cent one number is of another is found on page 117 of your textbook. Study the explanation on that page."

Later, as the pupils worked (on an individual basis), he said, "If there are questions, perhaps I can help you. When you finish reading, try exercises 1 and 2." (The textbook material is reproduced below.)

What Per Cent One Number Is of Another

The two finalists, Salem and Brockport, at a basketball tournament had these records. Salem had won 16 games and lost 5, and Brockport had won 18 and lost 6 games. Jack said Brockport was the better because they had won more games. Robert said, "That's wrong; you can't compare on games won. You have to find the per cent won." This is Robert's way of figuring:

```
         Salem                        Brockport
          .76 = 76%                    .75 = 75%
16/21 = 21)16.0              18/24 = 24)18.0
           14 7                         16 8
            1 30                         1 20
            1 26                         1 20
               4
```

Robert was correct, for Salem's per cent won was greater than Brockport's.

Robert had found what per cent one number is of another. For example, 3 is what per cent of 5? First we think, "What fraction?" It is $\frac{3}{5}$, but you know that $\frac{3}{5}$ of anything is the same as $\frac{6}{10}$, which is equal to $.6 = .60 = 60\%$.

Answer These Questions

1. 4 is what per cent of 8? of 12? of 20? of 50?
2. 6 is what per cent of 12? of 24? of 30? of 50?

To find what per cent one number is of another, first find what fraction it is, change the fraction to a decimal and then to per cent. For example, what per cent of 16 is 4? 4 is $\frac{4}{16}$ of 16; $\frac{4}{16} = \frac{1}{4} = .25 = 25\%$.

Fifteen examples to be solved followed this last solution.

As students studied the textbook material, the teacher helped various individuals who experienced difficulty. After about ten minutes the attention of the entire class was secured, and the answers that different individuals had obtained to the parts of exercises 1 and 2 were presented. Two of the solutions were explained by putting the computation on the board and permitting pupils to tell what they did.

At the conclusion of the explanations the teacher asked, "How do you find what per cent one number is of another?"

After a student had given the rule, all were directed to look at the statement in the book as one of the group read it.

The remainder of the period was spent doing the other fifteen examples.

In this upper-grade lesson, as in the lessons in the lower grades, a problem situation was used to initiate the work. Notice should be taken of the fact that the finding of per cents (the new thing to be studied) was identified in the first problem. That procedure is common in upper-grade instruction. Attention is called to the fact that questions which play such a major part in directing work in the lower grades are not used extensively in the lesson. That, too, is typical of upper-grade instruction. As has already been indicated, the presentation of complete solutions accompanied by a verbal explanation is the primary method of instruction.

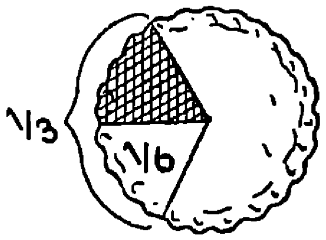
Sample Lesson 4. (Part of a lesson representing initial experience of a sixth-grade group with multiplication of a fraction by a fraction. The procedures are representative of type 2.)

"Today we are to study the multiplication of fractions," was the introductory statement used in this lesson. "Open your books to page 156 and read exercise 1." (Textbook page 156 and part of page 157 is reproduced below.)

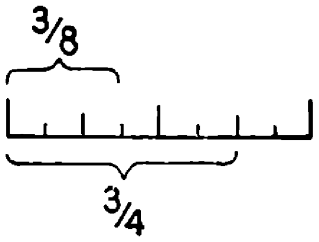
1. Nancy had a bar of candy which she shared with Beth. Each girl had what part of a bar? Then Mary came up and Nancy divided her part of the bar with her. What part of a bar did Mary get?

Did Nancy give Mary $\frac{1}{2}$ of $\frac{1}{2}$ of a bar? Did she give Mary $\frac{1}{4}$ of a bar? $\frac{1}{2}$ of $\frac{1}{2}$ = ?

2. What is $\frac{1}{2}$ of $\frac{1}{3}$ of a pie? The drawing shows that $\frac{1}{2}$ of a $\frac{1}{3}$ of a pie is $\frac{1}{6}$ of a pie.



3. What is $\frac{1}{2}$ of $\frac{3}{4}$ of an inch?



4. What is $\frac{1}{2}$ of $\frac{1}{4}$? Draw a circle to show that your answer is correct.

You remember that "of" means times, so $\frac{1}{2}$ of $\frac{1}{4}$ means $\frac{1}{2} \times \frac{1}{4}$. Example A shows how you can find $\frac{1}{2}$ of $\frac{1}{4}$ by multiplying.

A. $\frac{1}{2} \times \frac{1}{4} = \frac{1 \times 1}{2 \times 4} = \frac{1}{8}$
B. $\frac{1}{2} \times \frac{3}{5} = \frac{1 \times 3}{2 \times 5} = \frac{3}{10}$

First you multiply the numerators; $1 \times 1 = 1$. Write 1 for the numerator of your answer. Then you multiply the denominators; $2 \times 4 = 8$. Write 8 for the denominator of your answer. The answer is ———.

5. How was the numerator 3 obtained in Example *B*?

6. How was the denominator 10 obtained in Example *B*?

To multiply a fraction by a fraction, multiply the numerators to get a new numerator and the denominators to get a new denominator.

Copy and multiply:

a	*b*	*c*	*d*
$\frac{1}{4} \times \frac{1}{2}$	$\frac{2}{3} \times \frac{1}{2}$	$\frac{3}{4} \times \frac{1}{4}$	$\frac{1}{2} \times \frac{3}{4}$

After allowing a few minutes for reading exercise 1, the teacher asked a pupil, "What part of a bar did Beth get?" "What part did Mary get?" As the correct answers were given, these comments were made: "Yes, the drawing shows that when the bar was divided, each got a half." "Yes,

you can see from the drawing that when a half is divided, each piece is one-fourth of a whole bar."

The teacher then said, "Now look at exercise 2. It asks you to find what $\frac{1}{2}$ of $\frac{1}{3}$ of a pie is. What is it?" When the correct answer was given, attention was directed to exercise 3. After the answer to it was given, the teacher said, "Yes, the drawing shows that $\frac{3}{8}$ is $\frac{1}{2}$ of $\frac{3}{4}$. Now, let's look at exercise 4. It asks for $\frac{1}{2}$ of $\frac{1}{4}$. Look at the drawing of an inch and find $\frac{1}{4}$ on it. Two of the little marks, the eighths, make a fourth. What's $\frac{1}{2}$ of a $\frac{1}{4}$?"

When the answer was given, the teacher said, "Yes, it's an eighth. Now look at this circle (drawn on the board). This is $\frac{1}{4}$ of a circle. Now I have divided the $\frac{1}{4}$ into 2 parts. This is $\frac{1}{2}$ of a $\frac{1}{4}$. What part of a circle is it?"

Several children gave the correct answer. The explanation following exercise 4 and exercises 5 and 6 were then read by the teacher with only brief pauses for the pupils to answer the three questions. The teacher then said, "Let's spend the rest of our time multiplying the fractions in examples on page 157. Copy each example. Multiply numerators and then multiply denominators."

The remaining thirty-five minutes of the forty-minute period were spent in multiplying the fractions in the exercise. The teacher observed the work of various pupils and helped those who experienced difficulty.

In this lesson the developmental part of the work was hurried over. No discussion of any type was engaged in and no supplementary questions or study suggestions were made. Only the questions asked in the exercises were answered. The five minutes used in the development part of the lesson as compared with the thirty-five minutes allotted to computation is a partial illustration of the relative importance of the two phases in the thinking of this teacher.

Analysis of Instructional Procedures

The instructional procedures described under the headings of type 1 and type 2 and illustrated in the lessons represent only a small fraction of all the various teaching procedures that are used. While type 1 is generally considered superior, it falls far short of what many desire and advocate as superior teaching procedures. In a like manner the brief description of the type 2 teaching practices is far from representative of the truly inferior. However, an analysis of the trend of methods indicated in each procedure will help to identify important aspects of the teaching of arithmetic and should point to the reasons back of the varying practices which exist.

In type 1 procedures, emphasis is given to child participation. Thus, problem situations are used, both in the original setting and as the basis of the questions asked, to secure child participation. Use of such proce-

dures seems to be in harmony with the best principles of learning. A purpose is provided, one so immediate that children should easily identify it. Furthermore, participation by the child in the development of knowledges and skills makes for "identification with" and "ownership of." Both of these are considered important aspects of a good learning situation.

Another characteristic of type 1 method is that of developing various phases of processes or facts and having the pupil discover or see the relationship between the one studied and other processes or facts. While most textbooks and teachers favor direct telling and showing followed by demonstration by the pupil rather than provision of problem situations designed to direct discovery by the pupil, discovery, or search for, is an important part of initial learning.

In the presentation of instructional procedures, mention was made of the attempts to get pupils to arrive at generalizations. In the main, procedures on this point are far from the recommendations of learning theory. The generalization or rationalization in arithmetic is presented very soon after first experiences with the topics to which such generalizations apply. A look at the sample lessons will show that rules are given during the first lesson on a topic. Even if teachers prepare for the rule by leading questions and suggestions, generalizations developed so quickly are apt to be merely rules of thumb for the pupils and not true generalizations.

Recognition of variation in achievement is another point mentioned in the description of procedures which, in practice, lags far behind learning theory. For many years we have known that pupils differ in ability and in achievement in arithmetic, but for the most part nothing is done in classrooms to adjust the program to the needs of the different types of pupils. Sometimes the class is divided into many subgroups. Each subgroup is then given work supposedly suited to its ability. The net result is a teacher directing many classes, not one class, in arithmetic. Some groups receive more of the teacher's time than others, but, since there is only one teacher, no group receives adequate teacher guidance. Few teachers can successfully teach many subgroups, keep up class spirit, and have enough energy left for other teaching duties.

A more effective step toward meeting the problem of individual differences is observed when the teacher accepts each pupil's best effort as evidence of achievement, even though the actual performance seems cumbersome or immature. If we recognize that pupils can work at different levels in the same general area of arithmetic, part of the need for the numerous subgroups disappears. This attention to individual differences, which is a characteristic of the type 1 instructional practices, should make for continued improvement.

The breaking of phases of arithmetic into tiny bits, allotting the easier bits to the lower grades and postponing the more difficult phases, is another method of meeting individual differences. That this practice is wholly beneficial is in serious doubt. Consider, for example, the undesirability, from the standpoint of learning, of teaching multiplication by tens in one grade, by hundreds in the early part of the next, and by thousands still later, or the teaching of counting only to 10 or 20 in the first grade. When such arbitrary divisions are used, relationships within the number system cannot be effectively shown, and the advantage of logical organization of materials into learning units cannot be utilized. Furthermore, this breaking of the divisions of arithmetic into small parts does not provide instructional materials which present a challenge to the abler pupils. Consider, for example, the situation where a child spends a whole year dealing with sums not larger than 10. If the abler pupils are to be challenged, something more than the meager materials developed for the very poorest must be supplied. Since the addition of tens and hundreds is similar to the adding of ones, it is little short of stupidity to postpone for all pupils the presentation of addition of hundreds to Grade III or IV. Those teachers who are alert to the need for meeting individual differences provide through oral exercises and projects arithmetical experiences that go beyond the minimum set by textbooks and courses of study. Even though such practices are easily subject to abuses, teachers should be encouraged to use them.

The extensive use of objects, drawings, measures, and the like, in the lower grades is another characteristic of the type 1 instructional procedures. The primary purpose of the use of such visual learning aids is to demonstrate the truth and meaning of arithmetical facts or processes. No one questions the value of objects and drawings for demonstrating the truth of a fact or process, and there is general agreement that they aid in the development of some phases of the meaning of processes. However, use of them does not guarantee, and in some cases does not facilitate, the development of meaning. They are an intermediate step in the development, and, since they can be seen or manipulated, their use may do much to build up the child's confidence as well as his understanding. The use of such materials takes time, which is both an advantage and a disadvantage. It is an advantage in that it keeps teachers from plunging too rapidly into the computational phases of a topic. The disadvantage is that teachers are usually short of time, and, therefore, some other profitable activity must be omitted when time is given to work with objects, drawings, and the like.

The use of these special visual materials is far from standardized, and there is still much fumbling on the part of those who are using them.

Nevertheless, the value of visual materials is so apparent to all who use them that their use has markedly increased in the past few years. With increased use should come critical study and, eventually, a superior program.

In the description of type 1 instructional procedures, practice or drill was not emphasized. This lack of emphasis was due, in part, to difficulty of showing how it fits into the program and, in part, to wide divergence of views on the subject. Some teachers see no necessity for practice but depend entirely on use for mastery. Others are willing to devote the major part of instructional time to mastery exercises. It is generally agreed that practice for mastery on the important facts and processes is necessary if instruction is to meet the needs of all children.

The manner in which the practice for mastery work is to fit into the program is difficult to demonstrate. The general pattern is as follows: A few facts or a process may be presented in a problem situation. From this situation the facts are developed. There are experience exercises—such as simple problems and exercises in which the pupil shows the facts with objects or drawings, and then some practice exercises are provided. A few days later the same facts or processes appear again in practice exercises and again in the systematic reviews which are a part of most good programs of instruction.

From the standpoint of learning theory, the early phases of such a drill program are weak in that no provision is made to show the need for initial drill and that the early introduction of drill makes any kind of organization for efficient learning difficult. Study of textbooks and teaching practices shows that drill exercises are introduced without giving the pupil any reason for performing them. Perhaps the vaguely recognized assumption that all arithmetic must be practiced is sufficient. It seems rather strange, however, that special procedures (problems and projects) would be used to show that a process is needed and yet no reason is given for practicing for mastery. The importance of showing a reason for practice is still more evident when the effectiveness of the drill procedures which follow tests is considered. If the test reveals a lack of skill or knowledge, the pupil sees a reason for the practice which he is asked to do. It is obvious that the introduction of drill on facts before a sizable list of them has been developed makes organization for learning difficult.

The lack of emphasis on the setting of the problem and the development of the process is one of the points which attracts attention when the type 2 instructional procedure is contrasted with that of type 1. This failure to emphasize the setting or problem situation is based on the belief that interest in the setting detracts attention from the arithmetical process which is the important thing. There is in this criticism of problem-

settings a challenge which should not be ignored. Proponents of the use of problem-settings need to define their purpose as something more than the idea conveyed by the vague term, "setting." The verbal descriptions used are not true problems. If these situations are to stimulate research on the part of the pupils which eventually leads either to the discovery of that which is being taught or to the seeing of a need for it, then the place of problem-settings might be more easily defended. At present most of the problem-settings merely furnish a setting for textbooks or teachers to present and to explain a solution. Furthermore, since the problem-settings are presented for the purposes of illustrating the use of something new, then the solution of the problem through use of some other method seems appropriate. However, in very few classrooms are such solutions used.

In type 2 procedures little emphasis is given to developmental procedures. There is also some basis for lack of emphasis on developmental procedures, namely, the belief that analysis of and rationalizations concerning a process are most profitable after the student has some facility with and knowledge of the uses of a process. This tenet of type 2 procedures, like the matter of settings, should be a challenge to those who hold other views. It should be pointed out, however, that rationalizations receive little emphasis in the type 2 procedures, even after some facility in use has been acquired.

As has already been indicated, the instructional practices described under type 1 and type 2 in the preceding sections fall short of the ideal program as envisaged by some leaders in the field. Whether the methods proposed by such leaders are indicative of progress or whether these methods represent only the pet ideas of their creator is difficult to determine. Consideration of the procedures proposed by frontier thinkers in arithmetic methodology is made difficult by the fact that only fragmentary descriptions of actual practices are available. Since progress often results from study of and experimentation with the suggestions of frontier thinkers, attention is directed to a few of the new procedures.

In the opinion of the writer, the use of an extensive exploratory approach in the teaching of new processes is a procedure worthy of notice. While the exploratory exercises are directed toward the identification of the process to be taught, the process is not immediately identified or presented. In classrooms where these methods are employed, the pupil's first work in a new area of the subject usually consists of consideration of situations in which those competent in arithmetic would use the process that is to be taught. However, since the pupils do not know the process, they are not expected to use it. Suppose, for example, that the process of multiplication is being introduced. Situations involving multiplication

would be used, but no explanation of multiplication would be presented. Instead, pupils are encouraged to find the answers to the quantitative questions involved by addition, by doubling, by use of drawings, and by other means. The various ways of finding answers are then identified and an attempt is made to formulate a plan (rule) for dealing with such situations. In the formulation of this plan of solution or rule, multiplication situations are identified.

The finding and testing of the best method for solving such (multiplication) situations next becomes the assignment of the class. It is at this stage that the textbook or textbook plan for presentation of multiplication is frequently introduced. Notice should be taken of the fact that several multiplication situations have already been considered and the pupil is concerned with finding the best way of dealing with them. Pupils are encouraged to find out for themselves by actual test that the best method (book method) is better than other methods.

The division of a whole number by a fraction (e.g., $4 \div \frac{1}{2}$ and $5 \div \frac{2}{3}$) is a good phase of arithmetic to illustrate the procedure outlined in the preceding paragraph. Situations (usually verbal problems) in which pupils have to find the number of fractions in or equal to given whole numbers are presented for consideration. The task of the learners is to find the answers to the questions. They may do this by changing the whole number to the equivalent improper fraction and then dividing, by subtracting or by counting, by use of rulers marked in the appropriate fractions, by use of circles and other types of drawings, and the like. To insure use of some of these different ways of solving the problems, the teacher makes such suggestions as follows: "Show with a drawing that your answer is correct." "Use a ruler to test your answer." and "Try to find the answer to each question in at least two ways."

In the class discussion following the work described above, one of the items for consideration is the identification of the type of number situation illustrated in the problems that have just been solved. This may be done by considering something like the following: "What was it that we found in each of the problems? There was a whole number and a fraction in each one. What did we find?"

After the pupils have identified "dividing a whole number by a fraction" as the arithmetic process used in the solution of the problems, their next task is the finding or identification of a number solution for such situations. From their drawings and other solutions, the pupils can see that $4 \div \frac{1}{2} = 8$, $5 \div \frac{2}{3} = 7\frac{1}{2}$, and so on. The pupil's specific assignment may now be stated somewhat as follows: "What can be done with the figures in $4 \div \frac{1}{2}$ to get the quotient 8?" In the work period devoted to attempting to find an answer to the question above, this guide is followed:

"Test your solution on at least two situations." It should be noted that pupils have available the answers to several situations.

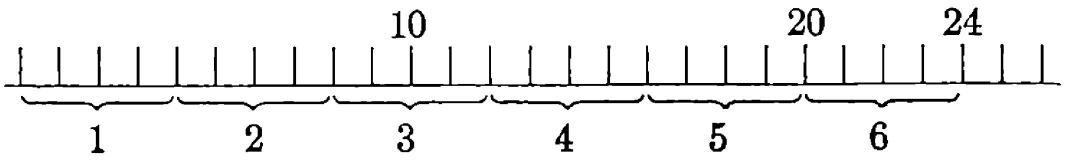
After a number solution for the division of a whole number by a fraction has been worked out, the pupils test their procedures on examples for which correct quotients are not available. Quotients are verified by means of drawings. The class then formulates an expression of the procedure. This expression is put on a chart or on the reference section of the blackboard and referred to when needed. Practice exercises to enable pupils to attain proficiency with the new procedure are then engaged in. A check on the pupil's understanding of the process is provided by means of a drawing which shows that answers to a few of the practice examples are correct.

Our Ways of Showing What Six Fours Equal

$\underbrace{4 + 4 + 4 + 4 + 4 + 4}_{\text{6 fours}} = 24$

1	2	3	4	5	6
1111	1111	11 | 11	1111	1111	1111

ten (1–3) ten (3–5) ones (6)



6 = 3 + 3
Then 6 × 4 is the same as
3 × 4 + 3 × 4 =
12 + 12 = 24

4 = 2 + 2. Then 6 × 4 is the same as
6 × 2 + 6 × 2 = 12 + 12 = 24

Proponents of the developmental type of procedure described in the preceding paragraphs claim that, by pupil participation in identification of processes and in the formulation of assignments, pupils acquire and exhibit an enthusiasm for learning in arithmetic similar to that which characterizes superior programs of instruction in social studies and science. Most students of methodology will agree that the question, "What can be done with the figures in 4 ÷ $\frac{1}{2}$ to get 8?" as that question is used in the procedure described, is far more challenging to the learner than is, "Look at the way 4 is divided by $\frac{1}{2}$ in the book (or at the board)," or similar conventional means of attracting the learner's interest. It is also claimed that by participation in the development of assignments the pupil has a better understanding of what his learning task is and that, because he helped in its formulation, he has a feeling of partnership, a desirable motivating factor. That these claims are plausible is readily

recognized. Whether actual practice will substantiate the claims remains to be seen.

The enrichment of various fundamental processes by means of pupil and teacher drawings, manipulation of objects, indirect and novel solutions, and the like, while not distinctly a method in the same sense as the developmental plan, has been emphasized so much by some leaders that attention should be called to the procedures here. Because the ways of enrichment of various arithmetic processes are so numerous and varied, space will not permit even brief coverage. As representative of one phase of this aspect of teaching, a part of one arithmetic bulletin board is reproduced on the preceding page.

Other Features of Learning and of Teaching Methods

The foregoing description of instructional procedures and the sample lessons shown indicate that the textbook has an important role in instruction. It serves in some instances as the only source of material for developing initial interest in a process; in others the beginning textbook exercises are used as a check, verification, or extension of ideas developed in class study. The textbook provides suggestions for development of processes, it provides explanation, along with study helps, practice exercises, and tests. In most classrooms the textbook is the course of study. That the modern text serves all these different uses adequately is undoubtedly saying too much. Arithmetic textbooks have changed greatly since the 1930 report of the National Society's Committee on Arithmetic. When that report was prepared the practice of a book for each grade was only beginning. At that time introductory procedures were not nearly so extensively developed as they are in present-day books. The same is true of some other features of books, notably the use of pictures, tests, and suggestions regarding interesting things to do.

The uses made of textbooks are obviously governed almost entirely by the teachers or the supervisory practices under which the teacher works. The sample lessons described in the preceding section show how teachers may supplement and enlarge the procedures suggested by the book or how the brief developmental procedure of the book can be practically by-passed. The best teachers use the textbook less than do inferior teachers. Two of the most important of such materials—visual aids and projects—have already been mentioned.

The uses of numbers in science, social-studies projects and activities, and other areas of children's schoolwork are other examples of material used instead of textbook material. The carrying out of some special project involving a great deal of computation is an especially popular way of supplementing the usual sources of arithmetic. The sales of Savings Stamps within schools, collecting and reporting scientific data such as

temperature, rainfall, silt carried by rivers, and the like, are examples of projects which may emphasize computation and interpretation of quantitative data.

The teachers who are probably the most successful organize arithmetic into units using textbooks as sources in much the same manner as the best sources are used in the study of social-studies or science problems. If a text presents a good explanation of a process, that text is used. Attention is called to the fact that in such uses of sources a need has to arise before there is a genuine motive for consulting the book. The unit must, therefore, provide situations which will show need for explanations, for practice exercises, for tests, and other similar things which textbooks can provide.

In the good arithmetic unit, the textbook is seldom used in the introduction to a new phase of arithmetic. A classroom situation in which class interest and participation are apparent is usually used in the introduction to new phases of work. The introduction in sample lesson 3 is an illustration of this type of work.

There are distinct advantages of using something other than the textbook situation for introducing new material. The chances are much better for interest, familiarity, and identification with a local situation than with a textbook situation. Furthermore, if interest in and some knowledge of a process has been acquired before turning to the book, there is a better possibility that the explanation accompanying the introduction given in the textbook will be understood. When there has already been some work with and discussion of a process, it is much easier to suggest reasons for consulting a book that contains useful materials. Such statements as, "Let's see if our textbook can help us on that," "Let's see whether this way we have worked out differs from that our textbook suggests," and "Our book has a short way of writing that. Let's look at it," are examples which show a suitable approach to the use of the textbook. With that type of approach, a challenging purpose is made apparent to the child. When such procedures are followed, the teaching of arithmetic becomes similar to the high level of teaching which is practiced in primary reading. In that field of study, the "Who-can-find-the-line-which-tells-what-Tom-did" type of suggestion has long ago supplanted the "Read-the-next-line" type of suggestion. The latter type, however, still prevails in many arithmetic classes.

While workbooks and worksheets are widely used, especially in schools representing the point of view illustrated by type 2 procedures, little in the way of superior instructional practices accompany this extensive use. One reason for this lack of contribution to improved instructional practices is that workbooks or worksheet exercises are often assigned to pupils because teachers do not know what else to do. It is not logical to assume

that such incompetent teachers will contribute much to teaching procedures. Busy work is seldom very profitable. Teachers who believe in the procedures where little time is given to development find that the exercises presented in the textbook are not sufficient to keep the pupils busy and therefore need the extra material which worksheets and workbooks provide. It is, therefore, the usual practice in such classrooms to give a daily assignment in the workbook. Again, it is not very logical to expect much in the way of improved instructional practices from those who believe that the chief way to teach is to provide all the work exercises pupils have time to do.

When properly used, however, workbooks are a valuable aid to the learner. For example, for some pupils who have poor work habits and depend upon other class members for most of their ideas, the assignment of exercises in a workbook is very beneficial. Then, too, there are occasions when the study of material in the textbook fails to result in the achievement desired. For such occasions the workbook supplies similar material that has not been studied. The workbook material may be used in a similar way with only part of a class that failed during textbook presentation to attain the level judged essential by the teacher. For all such situations the workbook supplies the specific material needed and thereby relieves the teacher of the burden of preparing such material. Attention is called to the fact that the uses just referred to are for specific purposes and not regular assignments just to give pupils extra practice or to keep them busy.

Use of Tests in Instruction

The use of tests in the teaching of arithmetic is an almost universal practice. From the beginning of instruction in this subject, the plan of "Teach (show), let the learner try or practice, and then find out how he is doing before teaching more or again" seems to have been used. At any rate, "Teach-test-teach" is, in spite of much criticism, still widely used. Even if teachers do not realize it, that is actually what is involved in much of their teaching. Many would say that the preceding statement is a bit exaggerated. However, practically all textbooks provide early in the book for each grade a series of exercises which are in fact, if not in name, tests designed to show what the child knows. As further evidence of the place of tests in beginning instruction, consider the following statement from a widely used book on the teaching of arithmetic: "It is essential, then, that the teacher begin the work of the year by studying individually the pupils of her class. The first step may well be the administration of group diagnostic tests."[1]

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[1] R. L. Morton, *The Teaching of Arithmetic in the Elementary School*, Vol. II, p. 44. New York: Silver Burdett Co., 1938.

Not only are tests used extensively at the beginning of a year's work but tests either in the form of everyday assignments or as regular test procedures are a part of instruction in every phase of the subject. The errors and lack of skill or knowledge revealed by tests are the basis for many of the explanations, demonstrations, and suggestions which teachers use in instruction. Diagnosis of pupils' achievement through observation of everyday procedures and other test situations is an important aspect of teaching and should be recognized as such. The taking of a test, the location of errors in the test, and the correction of such errors by the pupil are good learning experiences in themselves. Then, too, the use of tests is a powerful motivating force. It is only when time given to tests becomes excessive and when tests are used without other good instructional procedures that tests become detrimental to teaching. Like drill or practice, tests have a secure place in instructional procedures, and neither abuse through overemphasis by incompetent teachers nor severe criticism by critics has done much toward removing tests from classroom instructional practices.

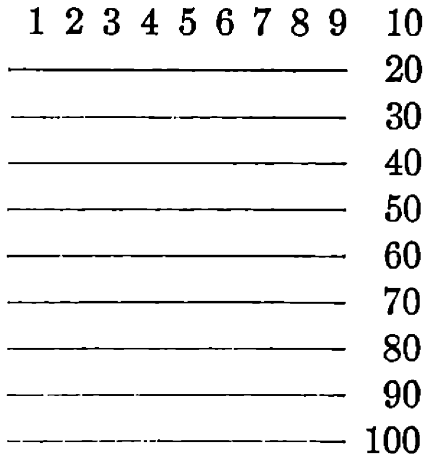
Special Learning Aids

The use of visual and other learning aids have in recent times become an important part of instructional procedure. The use of the blackboard as an aid to learning will serve as an illustration. Thirty years ago the amount of blackboard space in classrooms was much greater than it is today. At that time, pupils worked a great deal at the board, and observation of that work was undoubtedly an important part of teaching. While subject to misuse, such board work has definite advantages for the teacher. The work of many pupils can be observed very quickly and without much effort. Furthermore, many believe that the psychological effect on the pupils of doing work easily observed by others is beneficial. The teacher of today who wishes to observe the work of pupils must walk about the room and make his observations so conspicuously that often both pupils and teacher are ill at ease. As a result, teachers see less of the manner in which pupils actually work than is desirable for proper guidance of the pupils.

The representation of quantities through objects or counters is a procedure that is now widely used. Since some of the most useful number ideas a pupil possesses are based on thorough knowledge of the series of numbers to 10 and 100, special attention should be given to the counter type of learning aids which are used to represent 10 and 100. The most frequently used of these aids are in a linear form which insures easy identification of $\frac{1}{2}$'s, $\frac{1}{3}$'s, $\frac{1}{4}$'s, $\frac{1}{5}$'s and $\frac{1}{10}$'s. The use of linear form makes for the development of a linear mental picture, something very useful when such

frequently used terms as .37 or 37% are encountered. The superiority of the linear form over the rectangular (the other common form) is exemplified in the accompanying drawings.

This one example of the possible important role which special learning aids may play should be sufficient to call attention to the need for careful study of all such aids. It should be remembered that these are aids and that their place in instructional practices can only be based on their contribution to the major problems of arithmetic to which they pertain.



Problem-solving

The teaching of problem-solving receives much attention in the general literature and in discussions of arithmetic teaching, as well as in children's textbooks. In spite of this emphasis, there are no outstanding or easily recognized superior methods of teaching problem-solving. Most of the procedures emphasize careful reading and some of the steps associated with the formal steps in problem-solving. Estimating answers, selecting the process to be used, and stating the facts given are among the most popular suggestions. That such activities should improve the ability to solve verbal problems seems plausible. Teachers are not, however, satisfied with the results obtained and they are especially disturbed because many pupils have a dislike for this phase of arithmetic. Perhaps the difficulties encountered in teaching problem-solving are due to a fundamental error. Problems, as used in the modern arithmetic program, are, as Wheat and others have frequently pointed out, in most cases just exercises and not problems at all. The situations in science, social studies, and other areas of school life where numbers might aid in the solution of genuine problems are for the most part avoided.

In view of the conditions stated in the preceding paragraph, it is not surprising to find that emphasis on oral arithmetic, reading, and other procedures improves achievement in problem-solving about as much as practice with formal problem-solving exercises. There are even some teachers who claim that emphasis on number procedures and relationship within the system are good procedures for teaching problem-solving. Careful consideration of this last claim shows that it has much to commend it.

Theoretical and Experimental Considerations

The occasional references to learning theory in the preceding discussions imply the existence of principles and generalizations concerning learning in the field of arithmetic. There are such principles, but like most references to learning theory these are general and not specific. Examination of the educational literature reveals that there is frequent use of such terms as "new trends in learning theory," "new psychology," and the like. "Bond psychology," and "mechanistic psychology" are terms which are often used when reference is made to the old. Mere mention of 'old' and 'new' is not of great help to the reader, and one experiences great difficulty when attempting to find specific laws, principles, or generalizations concerning arithmetic learning. The most clearly discernible of the new trends are (*a*) the emphasis on goals in learning, (*b*) the recognition of insight, and (*c*) the reliance on organization and intrinsic relationships, which in turn imply larger blocks of material for study. In the first report of a National Society committee on arithmetic, the chapter on method places emphasis on minute analysis of subject matter resulting in identification of unit skills and larger hierarchies called abilities. This analytical approach led to reliance on studies of order of difficulty as a guide to instruction. Specific aspects of arithmetic, practically isolated from other aspects of the subject, received attention. Today, with the emphasis on securing a grasp of facts and processes, related facts and processes are considered without regard to order of difficulty.

While specific instructional practices today have their roots in psychology, it is very difficult to identify the psychological studies or theories on which any practice is based. This fact is not surprising, when one considers that most psychological studies of learning deal with the lower animals or very low-level human learning. Application of psychology based on such experimental evidence can hardly be rigidly applied to learning in such a highly organized field as arithmetic. However, in spite of such limitations, psychological theories in the form of principles of learning, and the like, have served as satisfactory guides in the development of instructional practices. As an example of the direct influence of psychological studies on practice, consider how closely practice is in agreement with the findings of studies in motivation, notably those concerned with knowledge of results. In addition to accepting the general laws of learning as guides, students of the teaching of arithmetic accept as guides most of the numbered statements appearing below. After each statement is a reference to supporting research or to a logical argument in favor of the principle of learning expressed by the statement.

1) The number system provides relations which are the best basis for understanding the major portions of arithmetic. The truth of this state-

ment is apparent to those who recognize and use the important aspects of our system in counting, in the adding, subtracting, multiplying, and dividing of tens, and so on. Katona's experiments[2] show that organization is an aid to learning and that organization based on intrinsic relations is the most effective.

2) The most effective learning procedures emphasize meaning and understanding. In the minds of those who have observed the daily work of pupils, there is no doubt about the value of using instructional practices which make for meaning and understanding. Controlled experiments which contrast meaningful and other procedures are difficult to perform. However, the clear superiority on delayed recall of meaningful teaching over mechanical teaching shown by one of Brownell's studies[3] is good evidence in support of meaningful teaching.

3) When directed by purpose, such processes as exploration, self-activity, and discovery, even though resembling trial and error, are valuable learning experiences. It is unthinkable that arithmetic, which is the foundation of the sciences, should not make use of these major characteristics of scientific method. Leaders have frequently suggested that pupils should not be given ready-made solutions and answers but should be encouraged to search for solutions. Studies by McConnell[4] and Thiele[5] have shown that such explorations have decided advantages for learning. The value of pupil investigation and discovery has long been considered significant. Herbert Spencer expressed his opinion with these words: "Children should be led to make their own investigations and to draw their own inferences. They should be told as little as possible and permitted to discover as much as possible."

4) It is advantageous to the learner to see the reasons for the uses or applications of the arithmetic that is being studied. That Americans have long had faith in this well-known guide to teaching is shown by a statement made by Baron Von Steuben: "The amazing thing about these people (Washington's recruits) is that you first have to tell them why, then how."

5) Generalizations should develop from experiences and in response to

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[2] G. Katona, *Organizing and Memorizing.* New York: Columbia University Press, 1940.

[3] W. A. Brownell, "An Experiment on Borrowing in Third-Grade Arithmetic, *Journal of Educational Research,* XLI (November, 1947), 161–71.

[4] T. R. McConnell, *Discovery versus Authoritative Identification.* University of Iowa Studies in Education, Vol. IX, No. 5, 1934.

[5] C. L. Thiele, *The Contribution of Generalization to the Learning of the Addition Facts.* Teachers College Contributions to Education, No. 763. New York: Bureau of Publications, Teachers College, Columbia University, 1938.

a clearly stated need. The truth of this statement seems so obvious that no supporting statements are needed.

6) Practice or drill on fundamental phases of arithmetic is essential if the needs of children are to be met. The saving in time and energy, to say nothing of the confidence generated, which automatic mastery of essential arithmetic affords is sufficient verification of this principle.

7) To be effective, drill should follow understanding. The detrimental effects[6] of premature drill support this statement.

8) Forgetting is normal, and, therefore, reviews and reteaching exercises are needed. Studies supporting the preceding are numerous and well known to students of learning.

9) Variation in achievement and rate of learning are characteristic of any group of pupils. This statement needs no supporting evidence.

Concluding Statement

This discussion of methods of teaching in arithmetic has dealt briefly with only parts of this broad field. Some parts, such as those involved in the sample reports of classroom-teaching situations, were overemphasized while other parts were either only mentioned or omitted. This mention or omission does not mean that such procedures are not important. There was not space for all. Regardless of omissions and possible misplaced emphasis, the presentation shows that arithmetic teaching involves many problems, that there are different views regarding method, and that apparently children learn (probably not too efficiently) by what many consider inferior methods. Among the many other conclusions that might be reached from a study of this presentation of methods, these two stand out: (*a*) It is the classroom activities, what the pupils and teacher do, which give character and zest to instructional procedures. (*b*) Some of the most popular instructional practices, to say nothing of the inferior ones, are not in quality the equal of methods in other fields in the elementary school, and some lag far behind learning theory. There is, then, room for much improvement.

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[6] W. A. Brownell, and C. B. Chazal, "The Effects of Premature Drill in Third-Grade Arithmetic," *Journal of Educational Research*, XXIX (September, 1935), 17–28.

CHAPTER VIII

THE PSYCHOLOGY OF LEARNING IN RELATION TO THE TEACHING OF ARITHMETIC

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Changing Concepts in the Psychology of Arithmetic

In each of three previous yearbooks dealing with arithmetic, special attention was given to the psychology of learning as it bears upon the teaching of arithmetic. The first[1] of these three yearbooks was published by the National Society for the Study of Education in 1930. While no single chapter carried the word "psychology" in its title, F. B. Knight's extended treatment of methods of teaching (pp. 145–267) was an application of the prevailing stimulus-response psychology to the learning of arithmetic. It is interesting to note that more than one-fourth of this chapter dealt with drill. In the following statement appearing in the introduction to the yearbook, Professor Knight characterized the general theory of learning assumed by the yearbook committee: "Theoretically, the main psychological basis is a behavioristic one, viewing skills and habits as fabrics of connections" (p. 5).

In 1935 the National Council of Teachers of Mathematics published a yearbook[2] entitled, *The Teaching of Arithmetic.* Chapter xii, which carried the title, "The New Psychology of Learning" was written by R. H. Wheeler. A radically different position was taken from that of Knight five years earlier. Wheeler accepted in full the Gestalt theory and advised teachers to "forget drills. Prepare your work logically and concentrate on relations" (p. 243). "The whole purpose of arithmetic is to discover number relationships and to be able to reason with numbers. It is not to learn the tables" (p. 247).

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[1] *Report of the Society's Committee on Arithmetic:* Part I, *Some Aspects of Modern Thought on Arithmetic;* Part II, *Research in Arithmetic.* Twenty-ninth Yearbook of the National Society for the Study of Education. Chicago: Distributed by the University of Chicago Press, 1930.

[2] *The Teaching of Arithmetic.* Tenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1935.

Six years later the National Council of Teachers of Mathematics published another yearbook[3] under the title, *Arithmetic in General Education.* In this volume, T. R. McConnell wrote the chapter entitled, "Recent Trends in Learning Theory: Their Application to the Psychology of Arithmetic." In this chapter (pp. 268–89), McConnell emphasized the place of organization in learning and the concept that learning is a meaningful process. His treatment is in marked contrast to that of Knight in 1930 and is more balanced than that of Wheeler in 1935.

The present chapter is directed to teachers and supervisors rather than to psychologists. The aim of the writer will be to make a sensible interpretation of psychological theory and experiments as they affect the arithmetic program. Illustrations will be drawn from the teaching of arithmetic to make the interpretations as clear as possible.

Much harm has been done to the organization and teaching of arithmetic by trying to force all learning situations to fit any one theory of learning. Wheeler's chapter in the 1935 yearbook, previously mentioned, is a good example of such an attempt. On one page Wheeler says to teachers, "Do not try to teach arithmetic; teach discovery, life, and nature through arithmetic."[4] On the following page he tells them, "Do not prepare pupils for an examination. Instead, teach arithmetic."[5] Even assuming that Wheeler could make concrete what he means by such a general phrase, his attempt in this instance to justify different teaching objectives in terms of a single theory of learning leaves the teacher in confusion as to whether he is or is not to teach arithmetic as such.

The very reason that there are conflicting theories of learning is that some theories seem to afford a better explanation of certain aspects or types of learning, while other theories stress the application of pertinent evidence and accepted principles to other aspects and types of learning. It should be remembered that the factual data on which all theories must be based are the same and are equally accessible to all psychologists. Theories grow and are popularized because of their particular value in explaining the facts, but they are not always applied with equal emphasis to the whole range of facts.

Classification of Theories of Learning

Since this chapter is directed to school teachers rather than to psychologists, no attempt will be made to deal with fine classifications of

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[3] *Arithmetic in General Education.* Sixteenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1941.

[4] *The Teaching of Arithmetic, op. cit.,* p. 243.

[5] *Ibid.,* p. 244.

learning theory. Two major theoretical positions only will be noted. These positions have been aptly distinguished by Hilgard.

The two main theories may be designated association theories, on the one hand, and field theories on the other. Any naming in this way does some violence to the individual theories, but nevertheless the typical American theories of functionalism, connectionism, and behaviorism have a common underlying logic which permits them to be grouped together, and the other theories, stemming chiefly from Gestalt psychology, have in turn a contrasting common ground. The theories here classified as association theories have been labeled reflex-arc theories and stimulus-response theories. The field theories group together various varieties of Gestalt, neo-Gestalt, organismic, or sign-significate theories.[6]

Influence of Association Theories

In the case of arithmetic, methods of organization and teaching have been affected by both the association theories and the field theories. Of the former, Thorndike's "connectionism" has played the principal role and, during the period when scientific methods were first applied to the study of school subjects, "connectionism" or "stimulus-response" psychology had great influence. It was particularly fitted to deal with the interests of that period. It was then that the relative difficulty of number combinations was being studied. It was in the early 1920's that the first analyses of textbooks showed the gross inequalities in frequency of occurrence of the different number facts. In the light of Thorndike's psychology and his statement of the laws of exercise and effect, arithmetic was a fertile field for applications. The rules governing drill were natural outcomes. Since each stimulus-response connection was believed to exist separately and independently, it was easy to draw the conclusion that mixed, unorganized drill was better than practice on a systematic arrangement of number facts. Devices such as the multiplication table were dropped; they emphasized system and did not fit into association theory as it was presented at that time.

It is true that a careful reading of Thorndike's psychology reveals an emphasis on the related and systematic character of arithmetic. For example, in one of his major writings on arithmetic he makes the following statement:

"Arithmetic consists, not of isolated, unrelated facts, but of parts of a total system, each part of which may help to knowledge of other parts, if it is learned properly. . . . Almost all arithmetical knowledge should be treated as an organized interrelated system."[7]

[6] Ernest R. Hilgard, *Theories of Learning*, p. 9. New York: Appleton-Century-Crofts, Inc., 1948.

[7] E. L. Thorndike, *New Methods in Arithmetic*, pp. 58–59. Chicago: Rand McNally & Co., 1921.

However, many students did not read Thorndike with that degree of care. They read his law of exercise and they read his statements regarding the nature of "bonds." They then provided the drills that they thought necessary to "fix" these bonds or connections. For example, Knight's emphasis on mixed drills did not lead to an appreciation of the systematic relationships among number facts. He emphasized the importance of fixing the number-fact connections through drills; and he had a very wide influence on the teaching of arithmetic. The fact that association theory had little to offer to problem-solving and to the understanding of arithmetical meanings disturbed few people because meanings had little place in the social-utility curriculum theory of the time, most of the emphasis being given to speed and accuracy in computation. Consequently, in the decade of the 1920's the most approved patterns of research were those which analyzed arithmetical processes into unit skills and then built up methods of teaching from these. The analysis and classification by one of Knight's students,[8] of all possible two-digit divisors, single-digit quotients, long-division examples, zeros excepted (there are 40,095 of them), is an illustration of the extent to which the emphasis on specific connections affected the teaching of arithmetic. It was quite natural that the number system and the arithmetical meanings of number operations were overlooked and that the main objectives were speed and accuracy in computation.

Influence of Field Theories

With the development of the "field theories" of learning, of which the Gestalt theory is most familiar to school teachers, the center of interest shifted from what was often, and perhaps unjustly, called an "atomistic" concept of learning to one which emphasized understanding of the number system and number relations and which stressed problem-solving more than drill on number facts and processes. The systematic character of the number system and the mental operations in problem-solving provided as attractive a field for the Gestalt psychologist as had the learning of number facts and computational processes for the association psychologist. Wheeler's chapter previously mentioned was only the beginning of the attempts to conceive of the teaching of arithmetic in Gestalt terms.

The field psychologist, of whatever school, thinks of his data in terms of the organization and systematic arrangement of the whole rather than in terms of elements set out in unrelated form. The concept of whole and parts is fundamental to his way of thinking. Wholes are organized structures of parts rather than simply a collection of parts. The whole is more, in significance, than the sum of the parts. A wrecked automobile is still

[8] Twenty-ninth Yearbook of the National Society for the Study of Education, *op. cit.*, p. 163.

all there, it weighs just as much, no bits are missing, but it will not function. All of the parts of a future house may be neatly piled on a vacant lot, but they do not make a house until they are put together in organized form. In a functional sense, a house is entirely different from the sum of its parts in unconstructed form. Likewise in arithmetic, a scrambled and unsystematic collection of digits is not a number system, nor are seven digits scattered at random on a page a number, although they may be systematically arranged to make any one of a group of numbers of which they are the parts.

The Role of Drill in Field Theory

There is, however, a place for practice and drill in arithmetic The distribution of number facts in drill must still be controlled carefully. But two distinct concepts have emerged which give drill a different position in the teaching of arithmetic. First, practice should follow, not precede, understanding. The old method of drilling on abstract number facts without a prior careful and concrete development of the facts is gone from a modern program of arithmetic. The function of practice is to increase efficiency of performance in operations which are already clearly understood. Such practice has an important place.

(The second concept is that effective drill must emphasize the systematic character of number relations and of the number system.) This means that the most effective practice will be so built that it will reinforce the systematic meanings that are learned in the other parts of the arithmetic program. The number tables, systematically arranged, which had dropped out of arithmetic textbooks are already reappearing, but with two modifications; first, they are not used until number meanings have been made clear through initial concrete teaching, and, second, they are appearing in varied forms, rather than in the single, stereotyped style of the old books. Drill on unorganized number facts contributed little to an understanding of the character of the number system. Properly organized drill can contribute to an understanding of the systematic arrangement of numbers.

Use of Number Charts. A simple counting table can help young children understand why any number can be written with only nine digits and a zero used as a placeholder. Such a table may be presented in the following form.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
.	.		.	.	.	.	.		
100	101	102	103	104	105	106	107	108	109
110	111	112	113	114	115	116	117	118	etc.

Pupils may then be asked such questions as: Why are the right-hand digits the same in each vertical column? Why are the right-hand digits in each horizontal row like the digits in the top row? Diagonal lines can be drawn to show other relationships. Study and practice with a counting table helps pupils to understand the orderly arrangement of numbers and the wholeness of a number system.

Knowledge of number combinations can be carried further through the use of tabular arrangements of numbers. Such tables can be made for each of the four fundamental processes. Those for addition and multiplication are shown below.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	0	18	27	36	45	54	63	72	81

A study of such number charts carries an understanding of number relations farther than is possible with a counting table. The relation of numbers horizontally, vertically, and along both diagonals reveals the systematic organization of number facts and gives a general understanding of the number system that no amount of drill on mixed-number combinations can give. Such number charts afford one of the best methods of verifying one's knowledge of number combinations.

Understanding Number Relations. The traditional number tables also illustrate the systematic character of the number system. They were abandoned because of poor methods of using them rather than because they did not possess potential values. If *studied* to show systematic relationships rather than simply drilled upon in sing-song fashion, they can be of service in increasing an understanding of number relations in addition to increasing one's familiarity with number combinations.

The distinction to be noted is that the earlier teaching methods emphasized drill on separate and unrelated number combinations, each to be learned as an independent connection. These drills resulted in a more or less complete mastery of number facts but laid no foundation for an understanding of number processes and operations. There was no attempt to generalize number relations. On the other hand, more recent teaching methods stressed the systematic and organized character of the number system and built up learnings which gave meaning to the four fundamental processes of dealing with numbers and to such operations

as borrowing and carrying. Furthermore, present learning theory stresses the importance of meanings throughout the whole range of number operations in addition to continued insistence on competence in computation.

Emphasis on Arithmetical Meanings

Types of Meaning Involved. The emphasis on meaningful arithmetic has created confusion among some as to just what kinds of meanings are intended. This confusion often has been due to the fact that some teachers have interpreted the phrase to imply social meanings and applications while others have had in mind mathematical meanings. Brownell has made a marked contribution toward clarifying this confusion by using the qualifying words, "meaning for" and "meaning of." He would classify as "meaning for" the applications of arithmetic to life needs which some writers have treated under the word "significance" as contrasted to "meaning." Brownell proposes that "meaningful arithmetic" is concerned primarily with "meanings of" arithmetic which may be called mathematical meanings. These must be taught so clearly that they will make sense to the children who learn them. Brownell then gives substance to his proposal by describing concretely some arithmetical meanings as follows:

> The meanings of arithmetic can be roughly grouped under a number of categories. I am suggesting four.
>
> 1. One group consists of a large list of basic concepts. Here, for example, are the meanings of whole numbers, of common fractions, of decimal fractions, of per cent, and, most persons would say, of ratio and proportion. Here belong, also, the denominate numbers, on which there is only slight disagreement concerning the particular units to be taught. Here, too, are the technical terms of arithmetic—addend, divisor, common denominator, and the like—and, again, there is some difference of opinion as to which terms are essential and which are not.
> 2. A second group of arithmetical meanings includes understanding of the fundamental operations. Children must know when to add, when to subtract, when to multiply, and when to divide. They must possess this knowledge, and they must also know what happens to the numbers used when a given operation is employed. If the newer textbooks afford trustworthy evidence on the point, the trend toward the teaching of the functions of the basic operations is well established. Few changes in the more recent textbooks, as compared with years ago, are more impressive.
> 3. A third group of meanings is composed of the more important principles, relationships, and generalizations of arithmetic, of which the following are typical: When 0 is added to a number, the value of that number is unchanged. The product of two abstract factors remains the same regardless of which factor is used as multiplier. The numerator and denominator of a fraction

may be divided by the same number without changing the value of the fraction.

4. A fourth group of meanings relates to the understanding of our decimal number system and its use in rationalizing our computational procedures and algorisms. There appears to be a growing tendency to devote more attention to the meanings of large numbers in terms of the place values of their digits. Likewise there is a strong tendency to rationalize the simpler computational operations, such as "carrying" in addition and "borrowing" in subtraction; but there is some hesitation about extending rationalizations very far into multiplication and division with whole numbers and fractions.[9]

This type of emphasis on the mathematical meanings in arithmetic is in accord with a psychology of learning which stresses relationships and organization in the content of education. It gives to the teacher of arithmetic goals of instruction which are very different from the limited goals of speed and accuracy which characterized arithmetic during the 1920's.

Organization of Content. The present emphasis on a more careful organization of the content of arithmetic reflects the influence of studies in the psychology of learning. Psychological experiments have repeatedly shown that organized content is more readily learned than is content presented in miscellaneous and unrelated fashion. The theory of incidental learning is defective chiefly on this account; it does not lead to an understanding of arithmetic as a whole. One of the main outcomes of the present emphasis on meaningful arithmetic has been an improvement in the sequence and organization of the content of arithmetic, as exhibited in courses of study and in textbooks. This improved organization is apparent in the relating of topics often treated as separate units in earlier courses and books.

An example of the tendency to organize materials into larger meaningful units is the treatment of fractions, decimals, and per cent as different ways of dealing with parts of wholes. The choice of which way to express a part, as a fraction, a decimal, or a per cent, is determined by common practice or by the apparent ease or convenience with which any one of the forms can be used for a particular purpose. All of the three ways are correct. Pupils are helped in their learning of arithmetic by an organization of teaching which shows the relatedness of these three processes. This emphasis on an organization which shows relatedness is helpful in small as well as in large topics. The topics of counting, adding, and multiplying can be organized to show that adding is, in one sense, a quick way of counting groups, and that multiplication is a still faster way of

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[9] William A. Brownell, "The Place of Meaning in the Teaching of Arithmetic," *Elementary School Journal,* XLVII (January, 1947), 257–58.

counting when the groups are of equal size. In all three cases the aim is to find a total. The treatment of ratio is another example of how a topic may be treated as something new and separate from what has previously been learned, or as simply another way of comparing numbers which, in some cases, makes the comparison easier to understand.

One of the great impediments to learning arithmetic has been the lack of meaningful organization of the content of arithmetic. A psychology which viewed learning as a sum total of many independently learned responses was not worried about the matter, but a psychological theory of learning which emphasizes the value of understanding places major emphasis on an organization of content and a selection of teaching methods which will make number relations more prominent and give meaning to the entire subject.

Motivation. Arithmetic, as taught during the 1920's, was so dominated by drill that it was monotonous and uninteresting to children. Motivation was mainly extrinsic in character, and the many devices that were used to add interest to a dull subject often had only slight relation to arithmetic. The idea was to sugar-coat the subject by attaching to it some extraneous activity in which pupils were thought to be interested. In the minds of the pupils, such extrinsic motivation made arithmetic appear as a necessary evil attached to the motivating device. It did not induce interest in arithmetic.

When arithmetic is taught meaningfully, it is interesting in its own right, and the need for extraneous motivation is reduced. Furthermore, until the child becomes interested in number relations as such, he will not go far in arithmetic. Even in the primary grades, arithmetic can be made interesting and popular by presenting it concretely so that number relations are really understood. This is now occurring in thousands of primary-grade classes that have adapted their methods and materials of instruction to the theory of learning based on getting meanings rather than to one which would emphasize practice on abstract number facts taught without any attempt to help children understand them. When pupils get an insight into the systematic character of the number system and the ways in which number processes actually carry on, the same interest appeals that operate in puzzles and card games are engendered, and the resulting satisfactions from success are similar. Teachers, who themselves understand arithmetic and who use ingenuity and imagination in presenting the subject meaningfully, find little need for extrinsic motivation.

Abstract and Concrete Materials

One of the troublesome problems for many teachers is how to teach the abstract generalizations of arithmetic. These occur at all levels from the

primary grades to the end of the elementary school. When association theory prevailed, the abstract-number generalizations were presented by the teacher as things to be learned through practice, and too little effort was made to help the pupil understand them. The child would memorize the facts, but it was generally only a verbal act which had little meaning to him. The field theories of learning have regarded understanding as of first-order importance and have insisted on an organization of materials that would make such understanding possible. Hence, good teaching begins with concrete situations that are full of meaning to the learner and gradually proceeds from the particular to the general so that when the abstract generalization is reached it is not a mere verbalism but is a statement full of meaning.

In the teaching of the abstract-number fact, $13 + 4 = 17$, good teaching begins with many varied examples, all within the experience of the learner and all subject to verification by counting. For example, 13 chairs + 4 chairs = 17 chairs; 13 books + 4 books = 17 books; and so on until finally the learner is convinced that 13 of anything + 4 of anything = 17 things and that always $13 + 4 = 17$. This procedure is in direct contrast to the older drill methods which *began* by drill on the abstractly stated number combinations without any preceding concrete experiences which would make the combination meaningful. The proper place of drill is only *after* the material drilled on is meaningful to the learner. When he then meets the combination $13 + 4 = 17$ often enough to learn it thoroughly he not only knows that 17 is correct but he understands why it must be the answer rather than some other number.

Abstract fraction combinations can be made meaningful by a gradual progression from concrete statements to the abstract number algorism. For example,

(*a*) 2 fifths-of-an-apple + 4 fifths-of-an-apple = 6 fifths-of-an-apple
2 fifths-of-a-pound + 4 fifths-of-a-pound = 6 fifths-of-a-pound

(*b*)
 2 fifths-of-a-mile
\+ 4 fifths-of-a-mile
 6 fifths-of-a-mile

(*c*)
 $\frac{2}{5}$ of a mile
\+ $\frac{4}{5}$ of a mile
 $\frac{6}{5}$ of a mile

(*d*)
 $\frac{2}{5}$ of anything
\+ $\frac{4}{5}$ of anything
 $\frac{6}{5}$ of anything

(*e*) $\frac{2}{5} + \frac{4}{5} = \frac{6}{5}$

When addition of fractions is presented in this way, adding fifths is no different from adding whole apples or whole numbers.

Recently there has developed a marked interest in improving the manipulative aids which serve to carry pupils from the concrete to the abstract levels of learning. The end product must always be the ability

to think and to operate abstractly. The detailed work on developing meanings has its function in the original process of providing understandings, but, once the understanding is clear, the operational process can gradually be stripped down to a skeleton of abstraction. However, using abstractions that have been made meaningful through concrete development is a very different thing from using abstractions that have been learned as verbal statements but have never been understood. A clear understanding of the psychology of generalizing from concrete experiences to abstract understandings is an essential of good teaching.

Factors Controlling Practice

As has already been said, teaching arithmetic with primary emphasis on meanings does not preclude the use of practice. It does change the position of practice, bringing it in only after the understandings concerned have been developed, and it also reduces the amount of practice needed. However, when practice is employed, two major contributions of psychology should be noted. First, the distribution of number combinations should be controlled and should be related to their difficulty. This means that there is still an obligation to see that all factors requiring practice are so distributed that they will be covered in the amounts needed and in proper position to support and maintain the learnings which have been achieved. One should remember also that difficulty is a function of the way number combinations are taught. The second matter of importance in teaching is that the length and distribution of practice periods should be managed with due regard for the findings of psychological studies of this problem. The length of practice periods needs to be adjusted to the age of the pupils, and the distribution of the periods must be related to the needs of learning in the first place and of maintaining learnings (reviews) after the initial learnings have been accomplished.

Effects of Maturation

During the last fifteen years teachers have become increasingly aware of the relation between the growth processes of the child and the learnings which are carried on in school. Olson's researches on the relation of growth to learning have important bearing on the time for carrying on different kinds of learning.[10] At present the effects of these studies are more noticeable in the point of view of teachers than in actual placement of topics or in the distribution of the load of learning, but as the relationships become more clear the implications for the teaching of arithmetic will be more apparent.

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[10] Willard C. Olson, *Child Development.* Boston: D. C. Heath & Co., 1949.

Conclusion

The teaching of arithmetic has been moving toward the developing of an understanding of number relations and of the number system as contrasted with thinking of arithmetic mainly in terms of separate combinations and operations. The outcomes of this emphasis go much beyond the computational efficiency which was almost the sole criterion of arithmetical ability in the 1920's. The contributions of a meaningful arithmetic to quantitative thinking have implications of first-order importance in a society such as ours.

CHAPTER IX

INSTRUCTIONAL MATERIALS FOR TEACHING ARITHMETIC

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I

RELATION OF INSTRUCTIONAL MATERIALS TO LEARNING

Definition of Terms

The success of a meaningful program in arithmetic depends, in a large degree, upon methods (see chap. vii) and materials of instruction. There is no one method or any single type of instructional material which will suffice in all situations. The skilful teacher selects methods and materials in terms of the outcomes to be achieved and of the needs and the interests of the children. If instruction in arithmetic is to insure a steady growth in understanding number relationships, a wide variety of instructional materials must be used to enrich and to supplement the learner's experiences.

Instructional Materials Include Anything Which Contributes to the Learning Process. Materials, as used in this chapter, include any picture, model, book, real activity, or teaching aid which provides experiences to the learner for purposes of (*a*) introducing, enriching, classifying, or summarizing abstract arithmetic concepts, (*b*) developing desirable attitudes toward arithmetic, and (*c*) stimulating further interest and activity on the part of the learner in the subject.

The value of instructional materials, the relative effectiveness of various materials, and the techniques of their use are frequently studied and discussed without consideration of their relationship to the fundamental

problem of learning. Such isolated discussions of instructional material may be so general as to be both vague and misleading. The uses of instructional materials can be scientifically investigated only when seen in their proper perspective.

Instructional materials need to be viewed both in terms of the principles of learning and the purposes of education. A rapidly growing body of research on learning emphasizes relatedness rather than itemization, and meaningful generalization instead of extreme specificity. This concept of learning as a process of growth rather than a result of mere repetition in experience regards learning as a meaningful process and considers understanding more important than mere repetition. It looks upon learning as a developmental process and encourages discovery and experimentation.[1] Learning of this kind does not occur quickly. Instead, many different kinds of experiences are needed for discovery and experimentation. To understand a program which encourages meaningful learnings, it is necessary to know the different levels or steps in the learning process and the kind of instructional material to use at each stage or level.

Steps in Learning Number

Learning number consists in an orderly series of experiences which begins with concrete objects and progresses toward abstractions. There are different stages, levels, or steps which may be identified in the learning process in arithmetic. These levels or steps are as follows:

1. Readiness for learning
2. Laboratory period for discovery
3. Verbal and symbolic representation of a quantitative situation
4. Systematic verbal presentation
5. Adult level of operation

There are no definite lines of demarcation among the different steps, but instead there may be overlapping between consecutive stages. The discussion which follows gives the role of each step in the learning of number.

Readiness for Learning

Readiness has often been considered synonymous with mental maturity and thus related to grade placement. The work of the Committee of Seven of the Superintendent's and Principal's Association of Northern Illinois is exemplary of this view. It is now recognized that readiness in arithmetic is a function not only of mental maturity and inner growth but also of previous experience, methods of learning, interests, attitudes,

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[1] A. I. Gates, A. T. Jersild, T. R. McConnell, and R. C. Challman, *Educational Psychology*, p. 344. New York: Macmillan Co., 1942.

and purposes.[2] Because of the complexity of the underlying factors, readiness cannot be delimited as to time or grade.

Brownell emphasizes the relatedness of ideas in arithmetic and gives a place to instructional influences as factors affecting readiness. He states that "a child is 'ready' to learn a new concept when he has control of all ideas and skills prerequisite thereto, when his previous experience has brought him to the stage when he can take on new learning."[3]

Motivation and purpose are important factors in developing readiness. McGeoch,[4] in discussing a typical process in learning, states that the first step in learning is a problem situation and that the problem situation results from a lack of adjustment between an organism's motivating needs, its immediate environment, and its reactive equipment. The number of problems of which a child is aware and the seriousness with which he responds to them are determined in considerable part by the extent of his information and experience in given fields.[5]

Readiness in arithmetic should be regarded as that period in the learning situation when the child's background for learning a new concept is appraised, the foundational experiences are provided, and a purpose for the new learning is established in the mind of the learner. One of the main tasks of the teacher during the period of readiness is to create a problem situation which will challenge the child's interest and which will provide the felt need basic to discovery and experimentation.

The Laboratory Method

The second step or stage in a meaningful program in arithmetic is a period of experimentation and discovery, when the pupil, through his own activity, searches for possible solutions to a problem. The pupil works with a variety of instructional materials. The exploratory activities which he goes through in seeking a solution to a new situation should result in the development of insights that enable him to make deductions and to formulate a tentative statement of the principles of operation in a particular situation. These tentative statements are applied and tested at later stages in the learning process and generalizations are evolved.

The laboratory period is not an activity period for the sake of having an activity. The teacher guides, questions, and stimulates the pupil to

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[2] *Ibid.*, p. 305.

[3] W. A. Brownell, "A Critique of the Committee of Seven's Investigations on Grade Placement of Arithmetic Topics," *Elementary School Journal*, XXXVIII (March, 1938), 505.

[4] J. A. McGeoch, *The Psychology of Human Learning*, p. 513. Chicago: Longmans, Green & Co., 1942.

[5] Gates *et al.*, *op. cit.*, p. 671.

make discoveries of principles and procedures by helping him use materials to portray a given situation. Equipment is needed for a laboratory in arithmetic just as in the physical sciences. Arithmetic is a science, just as physics is a science, and the steps of meaningful learning in arithmetic discussed here are an application of the method of science to arithmetic. Both arithmetic and the other sciences employ generalizations, and it should be no more difficult for a student in high school to understand Boyle's law which states that "the volume of a gas at constant temperature is inversely proportional to pressure" than it is for a pupil in the fourth or fifth grade to understand that the product of the denominators of two unlike fractions always gives a common denominator of the two fractions. In each case the statement is a generalization of experience. For effective learning of each generalization, the student should discover it for himself under the guidance of the teacher. The laboratory method is essential for this kind of learning.

There are two phases of the laboratory method which are important to help pupils make generalizations about number relationships and to understand methods of procedure for different processes. The first phase consists in individual discoveries. These occur when a pupil works independently under teacher guidance to find the answer to a given problem. For example, a student in the sixth grade may discover several different ways to find the perimeter of a rectangle 3 feet by 6 feet. He may find the number of feet in the perimeter as follows: $3 + 3 + 6 + 6 = 18$; or $2 \times 3 = 6$, $2 \times 6 = 12$, $6 + 12 = 18$; or $3 + 3 = 6$, $6 + 6 = 12$, $6 + 12 = 18$. The shortest way to find the perimeter is to substitute in the formula $p = 2\ (l + w)$. When individual discoveries are made in the classroom, the discoveries of all the pupils should be brought together by the teacher, placed on the blackboard, and discussed by the children. This discussion should lead to the formulation of one or more generalizations which are essential for effective work in the process.

The second phase of the laboratory period consists in group discoveries. In situations of this kind the class works as a group to make discoveries with manipulative or visual materials or with both. For example, a class in the third grade may work as a group with manipulative materials, such as markers or coins, to find how and why it is necessary to regroup numbers to subtract 18 from 43. In both phases of the laboratory period, demonstrations by a pupil, a group of pupils, or by the teacher may be part of the work of discovery. The demonstration, however, is not an initial part of the work, but this experimental procedure comes as a summary activity when the pupils are sharing and clarifying the discoveries both of individuals and of groups. Demonstrations used too early in the laboratory method result in imitative procedures and do not lead to the development of basic understandings.

Verbal and Symbolic Representation of an Experience

The third step in the sequence of presenting a new topic or process in arithmetic consists in making a verbal and a symbolic representation of an experience. First, the pupil makes an oral statement about the experience and then he uses numbers or other symbols to represent the experience.

We may illustrate the use of both verbal and symbolic representations of an experience by referring to a subtraction situation in which the class subtracts 18 from 43. After the pupil performs the experiment with manipulative materials, he makes a verbal description of the procedure. The import of his statement should be as follows: "43 is 4 tens and 3 ones, and 18 is 1 ten and 8 ones. It is not possible to take 8 ones from 3 ones. Take 1 ten from 4 tens, thus leaving 3 tens. Change the 1 ten to 10 ones, making 13 ones in all. 8 ones from 13 ones are 5 ones and 1 ten from 3 tens are 2 tens. The answer is 2 tens and 5 ones or 25."

After the pupil makes a verbal statement about the method of procedure, he makes a written record of the operation as shown at the right. Each successive step is at a higher level of development. The use of the crutch is permitted at this time because the aim of instruction now is understanding and not the development of efficiency in performing the subtraction algorism. After he understands the process, he should be encouraged to complete the example without the use of the crutch, as shown in the last illustration. The important part of the development in this step is to be able to state the procedure and to represent the operation with symbols.

$$\begin{array}{rcl} 43 & = & 4 \text{ tens } 3 \text{ ones} \\ -18 & = & 1 \text{ ten } \;\; 8 \text{ ones} \\ \hline \end{array}$$

$$\begin{array}{rcl} 43 & = & 3 \text{ tens } 13 \text{ ones} \\ -18 & = & 1 \text{ ten } \;\;\; 8 \text{ ones} \\ \hline & & 2 \text{ tens } \;\; 5 \text{ ones} \end{array}$$

$$\begin{array}{rr} {}^{3}\!\not{4} & {}^{1}3 \\ -1 & 8 \\ \hline 2 & 5 \end{array}$$

$$\begin{array}{r} 43 \\ -18 \\ \hline 25 \end{array}$$

Systematic Verbal Development of a Process

After the pupil makes a symbolic representation of an experience, he is ready to study the written development of the process in a textbook or a workbook. The development in a textbook is concise, and each statement is pregnant with meanings which the pupil cannot understand unless he has a rich experience before he reads the development. Frequently this background is missing, and, as a result, he reads meaningless statements.

If we pursue the development of compound subtraction, the textbook presentation of the example, $\begin{array}{r}43\\-18\\\hline\end{array}$, may be as follows: "43 is equal to 4 tens and 3 ones; 18 is equal to 1 ten and 8 ones. Since it is not possible

to take 8 ones from 3 ones, take 1 ten from 4 tens and change it to 10 ones, making 13 ones. Now subtract 8 ones from 13 ones and 1 ten from 3 tens." The pupil has just experienced the process described; therefore, he understands the meaning conveyed by the author of the text. Now the pupil is able to use the text as a reference book or a guide for his work. In case he wishes to refresh his understanding of any phase of the development, he is able to refer to the presentation given in the text. When the pupil does not have the background to interpret the development in a textbook, he memorizes the sequence of steps as shown in the model and then applies the techniques he saw to a new example. Imitation and not understanding is the basis of this kind of learning.

A textbook or a workbook presents a systematic treatment of number. In a sequential treatment of a topic there is a gradation of difficulty which is essential for the greatest ease in learning. Our educational program is so constructed that most learning results from the printed page. It is necessary, then, for a pupil to understand the systematic and sequential development of number as given in a textbook or some other book similarly constructed.

Adult Level of Performance

In initial learning the child uses objects to find a way to deal with number. He cannot deal with symbols, but he is able to move and manipulate objects. This is a low level of operation for dealing with quantities, but it is one at which the pupil understands what he is doing. The use of a crutch, likewise, may be essential in initial learning of a given process. At the adult level of performance, the pupil deals with symbols and not with things. He discards manipulative materials and crutches which were essential at a lower level of operation. Since individuals differ, it follows that pupils do not all reach the adult level of operation at the same time. There may be some learners in arithmetic who never are able to attain this level of operation.

We may be able to illustrate the adult level of operation by subtracting the example, $\begin{array}{r}43\\-18\\\hline\end{array}$. The thought pattern at this level is as follows: "8 from 13, 5; 1 from 3, 2." Not only does the pupil give the thought pattern unhesitatingly but also he knows with assurance that the answer 25 is correct. He has reached the stage which may be called "meaningful habituation." He is able to think intelligently about numbers. He may verify the result as follows: "18 is 2 less than 20; 20 from 43 is 23 and 2 more is 25." The pupil who is able to give a meaningful solution to an example or a problem has many facets of information which enable him to verify the result. Many pupils are able to solve the example at the right and find the correct answer, but they may not be able to $\begin{array}{r}\frac{1}{2}\\+\frac{1}{3}\\\hline\end{array}$

verify the result by intelligent thinking in number relations. These pupils solve problems mechanically and perform algorisms, but they are not able to interpret the results. A pupil who operates at an adult level of understanding of a process, such as adding the fractions $\frac{1}{2}$ and $\frac{1}{3}$, is able to verify the sum from his knowledge of fractions. He may state that $\frac{1}{2} + \frac{1}{2}$ is 1 whole and $\frac{1}{3} + \frac{1}{3}$ is $\frac{2}{3}$. Since the sum, $\frac{5}{6}$, is more than $\frac{2}{3}$ and less than 1 whole, the answer is sensible.

The pupil must achieve two things before he operates at the adult level of performance, which may be characterized as meaningful habituation. First, he must be able to give the solution in abstract form with accuracy and with a reasonable degree of speed. Second, he must be able to verify the result by making a reasonable approximation. The first of these results may be achieved by use of practice material, such as given in a textbook or a workbook. The second accomplishment results from a growing and expanding concept of number. This second phase of number receives a limited consideration in most arithmetic textbooks and in curriculums. The conventional procedure in a textbook is to rationalize a process, such as compound subtraction, and then to provide practice at successive grade levels for the pupil to achieve a mastery of the process. This plan is not conducive to growth in quantitative thinking, which is the second component of the adult level of performance. Johnson[6] proposed that the teaching of meaning should be spaced in the work of several grades just as occurs in such a topic as division of whole numbers. He suggested that the algorism should be taught first and then an enriched meaning of the process should follow at later grade levels. This plan minimizes the value of understanding when a process is first introduced. On the other hand, the plan he advocated calls for continuous enriching of the meaning of a topic at successive higher grade levels. This feature is important in achieving mastery at an adult level of performance in a given process.

The discussion of the five steps or levels in the learning process in number showed that materials are essential in the instructional program. We shall now consider the different kinds of materials and their uses at each stage in the learning process.

Classification of Instructional Material

As stated earlier, instructional materials include anything which contributes to the learning process. There are four kinds or classes of instructional materials: (*a*) *real experiences*, (*b*) *manipulative materials*, (*c*) *pictorial materials*, and (*d*) *symbolic materials*. This classification cannot be

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[6] J. T. Johnson, "What Do We Mean by Meaning in Arithmetic?" *Mathematics Teacher*, XL (December, 1948), 362–67.

regarded as rigid and inflexible. The various instructional aids overlap and blend into each other. The reader must view instructional material as existing on a continuum in which various materials appear in increasing abstraction as one proceeds from direct experiences to symbolic materials. The lowest level of quantitative thinking results from dealing with real objects, whereas the highest level results from dealing with abstract symbols.

Real experiences, considered by themselves, are not actually instructional material. They provide a medium through which objects and materials become available to children for careful study. Real experiences are treated here as instructional material because they make the real situation the learning situation in a functional setting. The things used or met in the experience, and not the experience itself, are the instructional material. The use made of the things determines whether or not the experience contributes to growth in understanding of number. The basis of all meaningful learning is purposeful, real experience. "Purposeful" because the learner must take an active part in the process of learning, and "real" in the sense that the learner is enabled to form clear and accurate concepts from the experience. A child cannot successfully interpret the symbols he sees in writing and in print unless he has accumulated a wealth of meaningful concepts and experiences with which to interpret the symbols. Consequently, anything which can be done to provide the child with real experiences will contribute to an adequate understanding in arithmetic.

Real experience, as used in this discussion, means tangible, direct, firsthand experience. Illustrations of such experience are counting the money in the milk fund, recording the time it takes to walk a city block, using a measuring cup to find that $\frac{1}{2}$ of $\frac{2}{3}$ is $\frac{1}{3}$, or making a deposit in a bank. Real experience means active participation in a real-life situation with responsibility for the outcome. Real, purposeful activities are means to an end and not an end in themselves. The desired end is growth in understanding and in ability to generalize about the situation.

Manipulative materials are those which the pupil is able to feel, touch, handle, and move. They may be real objects which have social applications in our everyday affairs, or they may be objects which are used to represent an idea or a characteristic of number or of the number system. Such objects as a measuring cup, ruler, scales, thermometer, and milk bottles of different size are used in our daily affairs. Materials designed specifically to help the pupil understand some phase of arithmetic include such objects as an abacus, factfinder, place-value pocket, markers, and fractional parts. The materials in the second group have little or no social significance, but this fact does not imply that these materials are less

valuable as instructional aids in arithmetic than those materials in the first group which have social significance and usage. An abacus shows how a blank rod may be used to represent a vacant position in a number. Zero performs this function in our number system. A pocket in a place-value pocket performs the same function as a rod on an abacus. Each rod or pocket may represent a place in the number system. Neither the abacus nor a place-value pocket has any social significance, but there is probably no better way to illustrate the meaning of place value and the function of zero as a place holder than by the use of these two instructional aids.

Pictorial materials include such things as charts, graphs, diagrams, and materials which may be projected on a screen. Slides, films, filmstrips, and opaque projector material also represent the kind of visual instructional material considered in this classification.

Symbolic materials constitute the fourth classification. The name implies that written or printed materials are in this group. The sources of symbolic materials are twofold: First, systematic study materials, such as textbook, workbook, and instructional tests; and second, quantitative situations which arise in other fields, as in science or in social studies. In most classrooms systematic study materials are the chief source of instructional material in arithmetic. To a large extent, the function of the other types of instructional material in arithmetic is to help the pupil operate at a high level of understanding with symbolic materials.

The Instructional Program

Since instructional materials are viewed in a broad sense to include anything which affects the learning process, the entire program of learning must be considered. There should be opportunities for excursions and field trips, as well as a place to use visual, manipulative, and symbolic materials. It follows, then, that the classroom must make provision for each of these kinds of instructional aids. Grossnickle[7] recommended that the classroom should become a laboratory where the pupil can manipulate things to discover arithmetic principles. The second step in the learning process previously discussed showed how the laboratory technique applies to the learning of arithmetic. On the other hand, an excursion may be the means to provide experiences for acquiring and learning number. Then, too, learning may result from the application of drill techniques; hence, the classroom must provide for this kind of learning. In short, the nature of a particular phase of the instructional program

[7] Foster E. Grossnickle, "The Use of Multi-sensory Aids in Developing Arithmetical Meanings," *Arithmetic, 1948*, pp. 1–14. Supplementary Educational Monographs, No. 66. Chicago: University of Chicago Press, 1948.

determines the usage of the classroom. Each kind of experience mentioned demands a different approach to the problem of learning number. It is not possible to evaluate the effectiveness of a total program of this kind by sampling a lesson at various intervals. It is possible to determine the quality of a bin of grain by examining samples from different parts of the bin because the grain is usually of the same structure. In case of a meaningful program in arithmetic, there is no one quality or trait which is descriptive of the entire program. Each of the five steps or levels in learning is different from the total program, yet these steps are interrelated. An effective program in arithmetic is one in which the proper instructional material is used at each stage in the learning process.

Summary

Part I of this discussion defined instructional material as anything which contributes to the learning process. The steps or stages in the learning of number are fivefold: (1) readiness; (2) application of the laboratory technique for exploration; (3) verbalization about a given experience and then representation of the experience with symbols; (4) systematic verbal presentation; and (5) mastery at the adult level of performance. The materials used at the various stages in the learning of number may be classified as: (1) real experiences; (2) manipulative; (3) pictorial and visual; and (4) symbolic. The use of materials at different levels of learning will be discussed in the following section of this chapter.

II
ILLUSTRATIONS OF THE USE OF INSTRUCTIONAL MATERIALS

We shall give illustrations to show how the use of instructional materials may serve to implement the steps in learning number.

Real Experiences

Teaching Ideas of Length (Readiness). "Miss X would like to have this table moved out of the room. Will it go through the door? How can we find out?" (Be sure that no standard units of measure are available in the room.) Encourage the children to suggest using string as one possible way to measure the table and the door, or using hand-spans, arm-lengths, pencil-lengths, book-lengths, or other nonstandard measures. This procedure shows the basic idea of measurement, which means that another thing may be substituted for the dimension of the real object.

Manipulative Materials

Real Objects

Teaching Ideas of Liquid Measure (Discovery). After an excursion to the creamery, the children asked questions about the sizes of various

milk bottles they saw. The teacher secured quart, pint, and half-pint bottles, as well as a measuring cup. The pupils first noticed the differences in sizes of bottles. Then their teacher said, "Each day you drink a half-pint of milk here at school. How many cups of milk do you think that is?" Several different answers were given. One of the group suggested that they find the correct answer by filling the half-pint bottle with water and then measure how many cupfuls it contains. The class followed this procedure in developing the idea that a quart is 2 pints or 4 cups, a pint is 2 cups or $\frac{1}{2}$ quart, and a half-pint is a cup. The children then made a picture record of what they had just discovered.

Using Objects to Represent an Idea

Number Grouping To Form Additional Facts (*Laboratory Technique*). Give each child 6 counting discs. Then ask him to see in what new and interesting ways he can arrange the discs in groups. Next have each child whisper something which he discovers independently. Select the answers which have something to do with the making of a number fact and ask each child who made the discovery to tell it to the class. Then the teacher has each pupil make the number combination with the discs.

Pictorial Materials

Charts

Teaching the Meaning of the Numbers from One to Ten. The children in the first grade made a number chart by placing colored paper balls in a pocket chart on the bulletin board. Then the children made a permanent chart of oaktag.

1	ONE	0									
2	TWO	0	0								
3	THREE	0	0	0							
4	FOUR	0	0	0	0						
5	FIVE	0	0	0	0	0					
6	SIX	0	0	0	0	0	0				
7	SEVEN	0	0	0	0	0	0	0			
8	EIGHT	0	0	0	0	0	0	0	0		
9	NINE	0	0	0	0	0	0	0	0	0	
10	TEN	0	0	0	0	0	0	0	0	0	0

Diagrams

Finding Half of a Group (*Verbalizing*). In a discussion centered around money, John said, "I earned 10 cents and I gave half of it to my sister, Mary." The teacher asked the class, "How much money did John have left?" "Who can make a drawing to show how much he had left?"

Several children volunteered, and they put the following diagrams on the board.

a) 0 0 0 0 0   0 0 0 0 0 = ten cents (10¢)
John gave away   John had left

b) 0 0 0 0 0 0 0 0 0 0 = ten cents (10¢)
0 0 0 0 0   0 0 0 0 0
Mary   John

c) [10¢] = [5¢] [5¢]
John   Mary

The group checked the diagrams and summarized by concluding, "One half of ten is five."

Making Addition Facts (*Symbolic Representation*). This example involves concrete objects, diagrams, and symbolism and illustrates a transition step. Each child counts six beads on a factfinder and arranges the six beads in as many different ways as he can. Each time he finds a new grouping, he makes a record of it on paper, using diagrams to make the record, as (000 000; 0 00000). Then the teacher has the different combinations written on the blackboard, as

	00		0	000
000	00	00	00000	00
000	00	0000		0

Now each pupil makes the combinations shown above. As he represents a combination on a factfinder, the teacher writes the corresponding numbers as follows:

000	3	00	2	00	2	0	1	000	3
000	3	00	2	0000	4	00000	5	00	2
	6	00	2		6		6	0	1
			6						6

Checking Understanding of Multiplication Facts (*Adult Mastery*). In an activity to check understanding of the multiplication facts, the teacher asked the pupil to give the answer to each fact and to *prove* his *answer.* He demonstrated the proof in a variety of ways as follows:

$$4 \times 5 = 20$$

a) 00000   00000   00000   00000

b) 0000   0000   0000   0000   0000

c) 11111   11111   11111   11111
5   10   15   20

d) 00000   10
00000
2 tens or 20
00000   10
00000

Projection Material

1. Filmstrips

The teacher of a third grade projected a number of frames from a filmstrip which showed how to do compound subtraction. A social situation arose in which it was necessary to subtract 35 from 52. Successive frames showed how to represent each amount with dimes and cents and then with markers. Since 5 cents cannot be taken from 2 cents, it is necessary to change a dime to 10 cents. The same process was portrayed with markers. Then the next frames showed the successive steps in subtracting cents from cents and dimes from dimes with the corresponding amounts in ones and tens. Finally, the pupils saw the completed solution. Then the teacher had the pupils perform the things they saw. The filmstrip served as a teacher's guide to introduce compound subtraction. The pupils did not learn how to subtract by seeing the filmstrip. They learned the process by experiencing the activities projected on the screen.

2. Flat Pictures

Teaching Comparison (Kdg. and Grade I, Step 1). Colorful pictures from magazines and story-books were used to teach the pupil an understanding of the words "big" and "little." He studied the pictures and indicated his understanding by locating things in the pictures according to their size. The type of question used is as follows:

1. Find the little girl.
2. Tell what the big boy is doing.
3. What color is the little dog?
4. How many big birds do you see?

Developing Ideas Relating to Time (Kdg., Step 1). Show pictures of typical daily activities (eating lunch, going to bed, rising, time for school, etc.) together with pictures of clocks showing the time for each of these activities.

Abstract Materials

Textbook

Teaching Addition of Tens and Ones (Steps 2 and 3). The teacher had the children read the first problem on page 20 to introduce the idea of adding tens and ones.

"John read 12 pages in his book yesterday and 17 pages today. How many pages did he read altogether?"

The children discussed the problem and decided that it is necessary to add to find the answer. They discussed the meaning of the numbers and gave the following solutions:

```
a)  1111111111      11
    1111111111      1111111
    2 tens and 9 ones = 29
b)    12          1 ten  and 2 ones
    +17           1 ten  and 7 ones
                  2 tens and 9 ones
c)    12
    +17
      29
```

The children discussed the solutions given and decided that adding tens and ones is like adding ones, but ones must be added to ones and tens to tens. The teacher said, "Now since we found how to add ones to ones and tens to tens, try out the rule by working the problems on page 20."

The previous illustrations were taken from part of the development of a particular phase of number learning in the kindergarten and primary grades. These illustrations were so varied as to show how each of the four major kinds of instructional material may be used in these grades. Now we shall describe learning situations in the upper grades and the kinds of material used.

Introducing Fractions (Grade V)

A fifth-grade class used objective materials in its study of fractions. The teacher had a set of fractional disks about 10 inches in diameter. One disk represented a whole and the other disks were divided, respectively, into halves, thirds, fourths, sixths, and eighths. These materials were for demonstration purposes. Each pupil had a corresponding set of fractional disks about 3 inches in diameter which were for individual experimentation.

Each pupil used his set of disks to show that two halves make a whole and that both halves are equal. Then he made a written record of the experience, as illustrated. In the same way the pupils used the other disks to show the components of one whole. Each pupil made a written record of his experience with different fractions.

$$\begin{array}{r} \frac{1}{2} \\ +\frac{1}{2} \\ \hline 1 \end{array}$$

Then the teacher had the pupils use other means to show the same things they demonstrated with these manufactured materials. Each pupil folded a sheet of paper to show halves, fourths, and eighths. He used his ruler to show the number of fractional parts in an inch. Finally, he made both circular and rectangular drawings to show the number of fractional parts in a whole.

The next step consisted in comparing fractions. The pupils used various fractional parts to compare fractions. The class arranged these fractions according to size in the following order: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$. Then the pupils discovered that the larger the denominator, the smaller the fraction. The

teacher had the pupils give fractions which meet certain conditions as to size, as, for example, a fraction that is greater than $\frac{1}{10}$ but less than $\frac{1}{8}$. In this way the pupils applied the principles learned about fractions from use of concrete material to fractions in the abstract form.

Then the teacher showed the film, *Parts of Things*. She directed the pupils to determine if the picture shows anything which they had not learned about fractions. The pupils enjoyed the film and some of their comments were as follows:

"I liked the part which shows that two halves must be alike."

"I liked the way the picture used milk bottles and melons to show halves and quarters."

"The picture did not show all of the fractions we made with our circles."

"I like the way the numerators and the denominators moved about in the picture. I'll never get them mixed up now."

These comments proved that the pupils had a background which enabled them to profit from the visual presentation. The usual sequence is to use the picture to create readiness and then to follow the visual presentation with the use of manipulative or symbolic material. In this case, the film presentation followed the use of manipulative materials. The picture gave an over-all summary of some of the points which the pupils already had learned about fractions. The film was a supplementary learning aid instead of a means to create readiness or to show the kind of experiences to provide the class.

Introducing Per cent (Grade VII)

The students of the seventh grade discussed the record of the local football team which won four of the seven games it played. The teacher asked how to write the record of the team. Nancy wrote the record as follows:

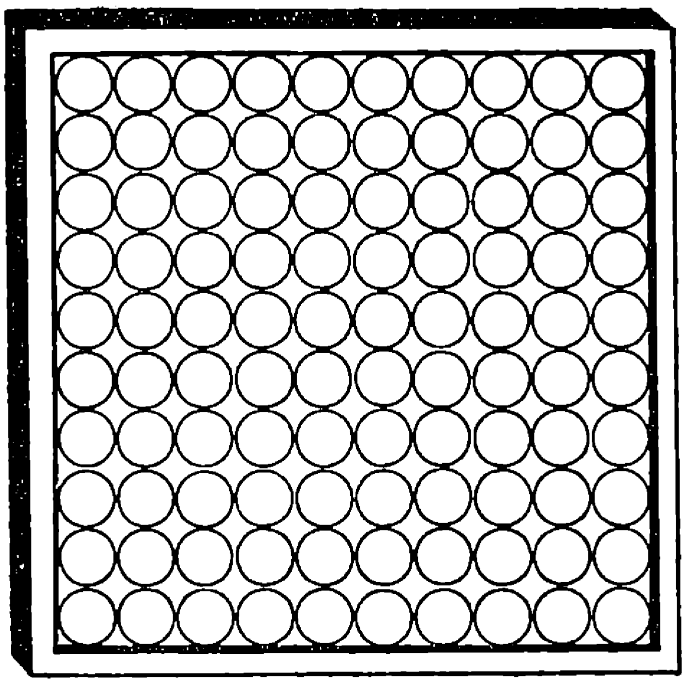
The team won 4 out of 7 games played.
The team lost 3 out of 7 games played.

Then the teacher asked if it is possible to write the record of the team in a shorter form. Bill wrote the following:

The team won $\frac{4}{7}$ of the games played.
The team lost $\frac{3}{7}$ of the games played.

Next, the teacher asked which of two teams had the better record if one team won $\frac{1}{2}$ of its games and another team won $\frac{3}{5}$ of its games. The class suggested that the two fractions should be reduced to a common denominator, as 10. Now it is easy to compare the two records since one team won $\frac{5}{10}$ of its games and the other team won $\frac{6}{10}$ of its games.

Finally, the teacher had the class decide which of two boys had the better record shooting fouls in basketball if one boy succeeded in 13 out of 20 "throws" while the other boy succeeded in 17 out of 25 throws. The class learned that it is advisable to reduce fractions to a common denominator in order to compare them. In this case, the first boy succeeded in $\frac{65}{100}$ of his throws and the other boy succeeded in $\frac{68}{100}$ of his throws. The teacher then told the class that each of these fractions may be written as a *per cent.*



Now she showed the class a *hundred-board.* She took one chip or disk from the board and stated that one disk represents 1 per cent. *One per cent (1%) means 1 from a group of 100.* She had the class show different per cents with the disks. The students soon were able to give the value of the fractions $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{10}$ in per cents. The students showed that six rows of 10 disks and 5 more disks represent the per cent of successful throws the one boy made and that six rows of disks plus 8 more disks represent the per cent of successful throws the other boy made. The second boy was successful in 3 per cent more throws than the first boy.

After the students were able to represent any per cent from 1 to 100 per cent with disks, the teacher asked if any one could show half a per cent. One student took a disk and broke it in half to show half a per cent. In the same way the class demonstrated how to represent $\frac{1}{4}$ per cent and $\frac{1}{8}$ per cent even though this was the first time the students studied per cent.

The next day each student ruled a piece of oaktag about 8 inches square into 100 squares. Then the teacher gave him 100 kernels of corn as used in Bingo. He used the corn to represent any per cent from 1 to 100 per cent. Likewise, he read different per cents from his cards when a certain number of squares were filled.

The next step in the demonstration consisted in finding a per cent of a number which is a multiple of 100. Thus, to find 3 per cent of 200, the thought pattern would be as follows: "3 per cent of 100 is 3. Since the number is 200, there will be two 3's, or 6." Likewise, the pupils could find 3% of 200 by using two 100-squares. Since 3% means 3 out of 100 on each board, there will be a total of 6 squares. The students objectified the amounts on their hundred-board of oaktag by placing 2 kernels in each square. Then they showed how to solve the example by use of common fractions, as $\frac{3}{100} \times 200 = 6$. After some practice in this form, the teacher asked if a different way could be used to write the common fraction. The class suggested the decimal notation and found that the answer obtained by use of decimals checked. At this point the teacher suggested that the students should consult the textbook. The text stated that *per cent means hundredths.*

The above discussion shows how different materials are used to introduce per cent in a meaningful program. The large hundred-board represents the kind of manipulative material to use in a laboratory for a class demonstration. The small hundred-board made of oaktag represents the kind of material to use in the laboratory period for pupil self-discovery or experimentation. Finally, the students reach the stage in which they discover, under teacher guidance, a quicker way to find a per cent of a number than can be found by use of manipulative material. These students have meaningful experiences which enable them to profit from the concise statements given in their textbook. The class is now ready to practice the exercises given in the textbook or workbook to attain a proficiency in the subject which should lead to an adult level of performance in finding a per cent of a number.

The reader found that instructional materials in arithmetic are classified as: (1) real experiences; (2) manipulative; (3) visual; and (4) symbolic. Many different illustrations showed the role of these materials in a meaningful program.

When To Use Objective Materials

The preceding section of this chapter showed how different kinds of materials are needed at various stages or steps in learning of number. The use of these materials must be co-ordinated and synchronized so as to form a unified program of instruction in arithmetic. These materials vary from the use of real experiences which are concrete to the use of symbols which are abstract. The question arises concerning the kind or kinds of materials to use with each pupil. It is important to know if all pupils are to begin a topic or process with manipulative materials or if it is possible for some pupils to begin with symbolic materials.

It is the function of good instruction in arithmetic to have a pupil operate at the highest level of difficulty at which the work is meaningful to him. It is obvious that there are pupils who develop sufficient insight into the meaning of number that the use of visual and manipulative materials is not needed to introduce a new process.

One of the writers observed a lesson in which the teacher introduced carrying in addition with the example at the right. He had the class identify the one's column and the ten's column. Four of the twenty-seven pupils in the third grade understood the reason for carrying. One pupil stated, "8 and 6 are 14 which is 1 ten and 4 ones. We write the 4 ones in the ones' column and we have a ten to add to the tens' column." This pupil had no need for manipulative materials because he could explain the sequence of steps in terms of the structure of the number system. On the other hand, the remaining twenty-three pupils did not understand the process, so they learned to carry by imitation. These pupils needed manipulative materials to objectify the regrouping which takes place when the 14 ones are changed to 1 ten and 4 ones.

$$\begin{array}{r} 36 \\ +18 \\ \hline \end{array}$$

There is a great need for reliable readiness tests which will indicate a pupil's understanding of various phases of number so that the teacher knows at what particular level this pupil should begin a new topic or process. In the absence of such tests, the teacher usually is correct to assume that all her pupils need to use manipulative material at the beginning of a new topic. It is important, however, for her to see that these materials are not used longer than is necessary for each pupil to discover the reason for a given procedure. As soon as a pupil is able to give a meaningful verbal statement about a solution, he should be encouraged to work with symbolic materials and not with objective materials. Individuals differ considerably in their ability to generalize and to work with abstract materials. There necessarily must be great variation in the length of time different pupils must use manipulative material before they are able to deal meaningfully with symbolic material.

III

VISUAL AND MANIPULATIVE MATERIALS

The final section of this discussion dealing with instructional materials gives a description of certain manipulative and visual aids to be used in the teaching of arithmetic. The writers have not attempted to give a complete list of materials now available. Instead, the items given represent a wide sampling of different types of instructional aids that are manufactured for use in the arithmetic classroom.

Visual and manipulative instructional aids are classified as: (*a*) manip-

ulative materials; (*b*) pictorial materials; and (*c*) projection materials. The latter classification includes any material suitable for projection on a screen.

Manipulative Materials

Manipulative objects and certain types of pictorial materials may be made in the classroom or purchased from supply houses. There is great educational value which comes from having the pupils make these materials. Sauble[8] has given a fine description of the kinds of material which the teacher is able to make and supply in order to enrich her instruction. The materials given in the following pages should be viewed as source material. Frequently a teacher has so many duties to perform that it is not possible for him to make needed supplementary materials. If he wishes to use the laboratory method with manufactured equipment, a wide variety of these materials is noted in the following pages.

1. *Concrete Materials.* Such concrete materials as 1-inch cubes, sticks, beads, clock faces with both Arabic and Roman numerals, and toy money are necessary items in an arithmetic laboratory. Materials of this kind are readily secured from most school-supply houses.

2. *Teach-a-Number Kit* (**22**).* The kit consists of a set of wooden blocks $1\frac{1}{2}'' \times 1\frac{3}{4}''$ in which a different colored block is used for each digit from one to ten. Each block is laoeled with the numeral and the word it represents. The blocks vary in thickness. The block to represent 1 is $\frac{1}{4}''$ thick and the block to represent each succeeding number increases $\frac{1}{4}''$ in thickness. Pupils may learn the value and meaning of numbers through comparison of sizes of blocks. Grade level: Principally the primary grades.

3. *Arithmetic Readiness Kit* (**13**). The kit consists of ten wooden blocks, $2'' \times 2'' \times \frac{3}{4}''$, with a hole through the center, a stringing lace, small individual number cards, and a small supply of play money. The materials may be used to develop the serial idea of numbers 1 to 10 and to show relationships among the basic coins. Grade level: I.

4. *Primary Number Cutouts* (**24**). These materials include a portable easel ($19'' \times 28''$) which has a display surface of black velour and an ample supply of cutouts of rabbits, ducks, stars, and circles that are lined on both sides with contrasting colored velour. The coherence of the velour surfaces holds the cutouts in position on the display board even when this board is almost vertical. The material is used to demonstrate different concepts, processes, and relationships with small groups. Grade level: Primary.

5. *Number Fact Finders* (**24**). Fact finders are designed for pupil use in count-

[8] Irene Sauble, "Enrichment of the Arithmetic Course: Utilizing Supplementary Materials and Devices," *Arithmetic in General Education*, pp. 157–95. Sixteenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1941.

* Numbers in parentheses refer to source of supply for visual and manipulative materials listed at the end of this chapter.

ing and in learning the number combinations. A fact finder consists of a number of movable beads on a wire rod. A fact finder for learning the concepts commonly taught in the first year is 8 inches long and has ten $\frac{1}{2}$-inch beads. The model for use in the second year is 10 inches long and has twenty $\frac{3}{8}$-inch beads.

6. *Number frames* (**24**). Number frames are designed for use by the teacher to demonstrate counting and to illustrate number groups. These aids are enlarged models of fact finders.

7. *The Educator Number Fence* (**11**). A number fence consists of a board drilled for the insertion of 10 pegs in a single row. Slots are provided between the pegs so that cardboard fences may be placed to form separate groups. The board is designed to enable pupils to discover addition and subtraction facts. Grade level: Primary.

8. *Counting Disks* (**24**). These disks are made of durable fiber about $1\frac{1}{4}''$ in diameter in a solid color. They lend themselves to counting, discovering number facts, and to giving demonstrations of the fundamental operations as far as three-place numbers. Grade level: I to VI.

9. *The Twenty Board* (**24**). This board consists of a framed cardboard rectangle large enough to hold 20 cardboard disks in 2 rows of 10 disks each. Its chief use is to illustrate the addition facts with sums in the teens and the corresponding subtraction facts. Grade level: Primary.

10. *Modernized Abacus* (**24**). This abacus has a solid backing which supports 4 vertical wires with different-colored beads to represent the first four places in our number system. Nine beads of the same color are on each wire with a top tenth bead of the next succeeding color. The tenth bead completes the ten and shows by its color that 10 beads are equal to 1 bead of the next higher order. Grade level: I to IX.

11. *Ten-Ten Counting Frame* (**12**). This counting frame, $8\frac{1}{2}'' \times 9''$, is made of wood and it supports ten wires, each of which holds ten $\frac{1}{2}''$ beads. Its principal uses are to provide a counting activity and to emphasize the fact that our number system is a system of tens. Grade level: Primary.

12. *The Hundred-Board* (**24**). The hundred-board consists of a framed cardboard 20 inches square equipped with 100 cardboard disks and two cards 20 inches square. The one card is a counting card which contains the numbers 1 to 100 printed in sequence, ten numbers to a line. The other card is a product card and it shows the multiplication products to 10×10. The hundred-board is designed for demonstration purposes to teach counting, to develop the basic facts meaningfully in all four processes, as well as to develop some of the concepts, processes, and relationships that are involved in decimal fractions and in per cent. Grade level: I to VIII.

13. *Place-Value Pockets* (**24**). Place-value pockets consist of a wooden frame 24 inches long to which are attached three wooden pockets labeled "Hundreds," "Tens," and "Ones," respectively, for inserting markers. It may be used to demonstrate place value, to illustrate carrying in addition and multiplication, to show transformation in subtraction, and to give meaning to placement of quotient in division. Grade level: II to VI.

14. *Fractional Parts* (**24**). A set of fractional parts consists of forty-nine

pieces: twenty 1-inch squares, and seven 4-inch disks, including one whole disk and six which are dissected, respectively, into halves, fourths, eighths, thirds, sixths, and fifths. The squares and disks are die stamped on a single sheet of heavy cardboard with face and reverse side of contrasting colors. Each disk section has its size imprinted on its face. The set is designed for pupil use and provides the materials necessary for a meaningful study of fractions. Grade level: IV to IX.

15. *Fractions Made Easy* (6). This set of materials includes twelve cards, each 4 inches square, on which are printed basic units in the shape of squares, triangles, and circles. These units are divided into equal parts representing halves, fourths, eighths, sixteenths, thirds, sixths, and twelfths. Cutouts of different colors, but corresponding to the shapes illustrated on the unit cards, are used by the pupils to discover fractional relationships. Grade level: IV to VI.

16. *Fractional Parts Enlarged* (24). A set of these materials consists of a display board, 19″ × 28″, which is lined on one side with black velour, and an assortment of 8-inch disks, dissected disks, and 2-inch squares. These squares and disks are lined on both sides with velour paper in contrasting bright colors. The disks and squares adhere firmly when placed on the display board which can be held in almost a vertical position. These disks provide a complete set of materials which the teacher may use to demonstrate the concept of a fraction, fractional relationships, and all operations with common fractions. Grade level: IV to IX.

17. *Exton's Parts-Imparter* (19). This set of materials includes a 9-inch "Double Disc" and two "Equivalence Wall Charts" for teacher use and classroom demonstration. There is available for the pupil a $4\frac{1}{2}$″ "Double Disc" which corresponds to the teacher's model. These materials are used to teach parts of a whole, fractions, percentage, decimals, angles, degrees, and equivalence. Grade level: IV to IX.

18. *Fraction Wheel* (6). This set of disks has ten 7-inch circles which have been segmented by die cutting to show halves, fourths, eighths, sixteenths, thirds, sixths, ninths, twelfths, fifths, and tenths. The kit box has a self-contained easel so that materials may be used for classroom demonstration. Grade level: IV to IX.

19. *Fraction Chart* (24). This chart consists of a 21-inch square blackboard, on which are fastened 6 slides. Die-stamped cards representing fractional parts of one whole may be fitted into each of these slides. The chart is designed to stand vertically. The fraction cards furnish a basis for teaching the concept of fractional parts of a unit, for changing fractions to higher and lower terms, and for adding and subtracting fractions. Grade level: IV to IX.

20. *Moto-Math Set* (25). This set consists of a blackboard graph chart, 32″ × 34″, made of 20-gauge steel plate with 977 holes $\frac{1}{16}$″ in diameter, and various accessories which may be quickly mounted on the board by means of split pins. The accessories which may be used in the teaching of arithmetic include (*a*) counting disks of hard vulcanized white fiber $1\frac{1}{2}$″ in diameter and $\frac{1}{16}$″ thick with a $\frac{1}{16}$″ hole in the center for a split pin with which to mount them on the graph chart in any desired arrangement; (*b*) fraction disks which are 12″ in diameter,

one of red fiber and the other white. This material is useful in teaching fractional parts of a unit; and (*c*) the abacus which consists of three strands of ten large beads each, finished in bright red, blue, and yellow enamel (a special spring holds each bead in any desired position on the wires when mounted vertically). Grade level: I to IX.

Pictorial Materials

The following pictorial materials have been selected because they are designed to help the teacher provide a meaningful approach to an understanding of the number system. Flash cards and other materials which are usually associated with a drill program for teaching the basic number facts have been omitted. These materials, however, are readily obtained from school-supply houses. On the other hand, newer types of perception cards have been included where such cards will aid in teaching arithmetc meaningfully.

1. *Number Readiness Charts* (**18**). The set consists of fourteen separate charts $18\frac{1}{2}'' \times 24\frac{1}{2}''$, and one perforated sheet of 67 cutouts. The charts are colored pictures representing experiences familiar to young children. Slots are cut in the charts so that the cutouts may be inserted. The set of charts is designed for use at the beginning of the first year of instruction to provide a series of experiences that should precede the usual systematic program.

2. *Class Number Chart* (**5**). This chart shows pictures, numerals, and names of numbers from 1 to 10 and is printed on a large cardboard for classroom display. The chart is designed to teach the reading and the writing of numerals. Grade level: I.

3. *The 100-Chart and the 200-Chart* (**20**). The 100-chart consists of 10 strips $3'' \times 22''$. A row of 10 red dots is printed on each strip. A wall chart is required to display the materials properly. The 200-chart includes all the material listed for the 100-chart and supplies, in addition, a chart with 100 red dots arranged in rows of ten. These charts are valuable in teaching the meaning of two-place and three-place numbers. Grade level: I to IV.

4. *Counting Chart* (**5**). This chart provides symbols in semi-concrete form which may be used for counting from 1 to 100. Grade level: I to II.

5. *One Hundred Chart* (**12**). The chart, $25'' \times 29''$, is printed on heavy paper and is designed for classroom use. The main purpose of this material is to help teach counting to 100 and to visualize the number symbols in the process. Grade level: I to III.

6. *International Metric System Chart* (**21**). This chart shows the comparison between the English and Metric systems of measurement for length, area, volume, liquid and dry measure. The chart measures $28'' \times 44''$, and all illustrations are full size. Grade level: V to IX.

7. *Bulletin Board Charts on Arithmetic* (**14**). These charts, $10'' \times 13''$, are printed on light cardboard of attractive shades. Twenty different posters are included in a set and are titled: "Measures I," "Measures II," "A Time Chart," "Linear Measure," "Roman Numerals," "Change for a Dollar," "Kinds of

Subtraction," "Temperature," "Weight," "Liquid and Dry Measure," "Measurement by Counting," "Decimals," "Using Decimals I," "Using Decimals II," "A Fraction Chart," "More Fractions," "Adding Fractions," "Interest Problems," "A Long Division Chart," "Percentage Equivalents I," and "Percentage Equivalents II." Grade level: I to IX.

8. *Numbers We See* (**18**). This is a 72-page book of pictures in color and designed to provide a systematic development of number concepts by nonformal, concrete methods. Although this is a bound book, it is included in this listing because it is entirely pictorial. A teacher's edition is available which gives detailed suggestions for using each page. Grade level: I.

9. *Picture-Symbol Cards* (**5, 9, 12, 16, 20**). These cards are used primarily to teach the number symbol by associating it with the picture of the group of objects it represents. Sets of cards for numbers 1 to 10 are available in sizes suitable for pupil use and class use. Grade level: Primary.

10. *Group Recognition Cards* (**5, 6, 9, 18, 20**). Cards of various sizes show number groups either by using pictures of real objects or by using dots, squares, triangles, or lines arranged in patterns to represent the number groups. Sets are available for individual pupil use and class use. Grade level: Primary.

11. *Fraction Cards* (**20**). This set of ten cards, $5\frac{1}{2}'' \times 6''$, pictures units in the shape of squares, triangles, and circles divided to represent halves, fourths, and thirds. These cards are suitable for class use in teaching the fraction concept. Grade level: II to IV.

Projection Materials

Projection materials are listed under the following headings: (*a*) basic number concepts; (*b*) whole numbers; (*c*) fractions; (*d*) decimals; (*e*) per cent; (*f*) measurement; and (*g*) social applications. Each film and filmstrip in the following discussion is evaluated. The appraisal represents the composite group judgment of a committee of classroom teachers at the grade level for which the film is designated. The evaluation is based on the suitability of the material as a teaching instrument to accomplish the purposes indicated by the producers at a specified grade level.

Basic Number Concepts

1. *Let's Count* (**1**). One reel, 16-mm. sound film; black and white and in color. Collaborator: F. Lynwood Wren. Grade level: Primary. This film is valuable to the teacher in suggesting types of counting activities which may be used in the classroom. Teachers' guide available. Evaluation: Good.

2. *Using Numbers* (**2**). A series of sixteen 35-mm. filmstrips; black and white pictures and drawings. Collaborators: John R. Clark and Caroline H. Clark. Grade level: Primary. These filmstrips may be used in conjunction with class instruction in developing an understanding of the meaning, sequence, use, and writing of numbers. Careful preparation on the part of the teacher before she uses each filmstrip is essential. These visual aids are excellent for review pur-

poses. No teachers' guide is available. Evaluation: Good. The titles of the different filmstrips are as follows:

"Counting to 5"	"Counting by 10's to 30"
"Counting to 10"	"Conting by 10's to 50"
"Reading Numbers to 10"	"Counting by 10's to 80"
"Writing Numbers to 10"	"Counting by 10's to 100"
"Counting from 10 to 15"	"Reading Numbers to 50"
"Counting from 15 to 20"	"Reading Numbers to 100"
"Counting from 20 to 40"	"Working with Numbers to 100"
"Counting from 40 to 100"	"Writing Numbers to 100"

3. *What Is Four?* (26). One and one-half reels, 16-mm. sound film; black and white. Advisers: William A. Brownell and Laura K. Eads. Grade level: Primary. The large number of concepts discussed in this film would make it more suitable for review purposes than for introductory procedures. It is, however, excellent for teacher-training purposes. Teachers' guide available. Evaluation: Good.

4. *What Numbers Mean* (17). A 35-mm. filmstrip, 37 frames; in color. Adviser: Foster E. Grosnickle. Grade level: Primary. This filmstrip shows the meaning of each number from one to ten, inclusive, in concrete form and then in semi-concrete form. Teachers' Guide available. Evaluation: Excellent.

5. *The Teen Numbers* (26). One reel, 16-mm. sound film; black and white. Advisers: William A. Brownell and Laura K. Eads. Grade level: II to III. This is an excellent film to supplement classroom instruction where children have abundant opportunity to manipulate objective material. Teachers' guide available. Evaluation: Excellent.

6. *Zero a Place-Holder* (17). A 35-mm. filmstrip; in color. Adviser: Foster E. Grossnickle. Grade level: Primary. This film provides an excellent pattern for teachers to follow in using manipulative material. Teachers' guide available. Evaluation: Excellent.

Whole Numbers

1. *Parts of Nine* (26). One reel, 16-mm. sound film; black and white. Advisers: William A. Brownell and Laura K. Eads. Grade level: I to II. The film presents the meaning of nine and the addition and subtraction facts involved in nine. Teachers' guide available. Evaluation: Very good.

2. *Addition Is Easy* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: F. Lynwood Wren. Grade level: III to IV. The film develops the basic concept of column addition with carrying. Teachers' guide available. Evaluation: Good.

3. *A Number Family in Addition* (17). A 35-mm. filmstrip, 37 frames in color. Adviser: Foster E. Grossnickle. Grade level: Primary. This film shows teachers how to present a basic number fact in addition. Teachers' guide available. Evaluation: Excellent.

4. *Subtraction Is Easy* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: F. Lynwood Wren. Grade level: III to IV. The film is suitable to accompany classroom instruction in the fundamentals of subtraction, transformation of a number, and place value. Teachers' guide available. Evaluation: Good.

5. *Borrowing in Subtraction* (23). One reel, 16-mm. sound film; black and white. Grade level: III to V. The picture presents concretely the need and the principle of transformation in compound subtraction. It is excellent for review purposes. Teachers' guide available. Evaluation: Excellent.

6. *Compound Subtraction* (17). A 35-mm. filmstrip, 36 frames in color. Adviser: Foster E. Grossnickle. Grade level: Primary. This film shows how to introduce the "decomposition method of subtraction." Teachers' guide available. Evaluation: Excellent.

7. *Addition and Subtraction* (7). A 35-mm. filmstrip, 30 frames; black and white. Grade level: X and up. This filmstrip is suitable for a refresher course. Teachers' guide available. Evaluation: Good.

8. *The Threes* (17). A 35-mm. filmstrip, 40 frames in color. Adviser: Foster E. Grossnickle. Grade level: Primary. This filmstrip shows the development of the threes in multiplication. Teachers' guide available. Evaluation: Very good.

9. *The Twos in Division* (17). A 35-mm. filmstrip, 46 frames in color. Adviser: Foster E. Grossnickle. Grade level: II to III. The filmstrip develops the entire table of twos in division. Teachers' guide available. Evaluation: Good.

10. *Meaning of Long Division* (2). One reel, 16-mm. sound film; black and white and in color. Collaborator: M. L. Hartung. Grade level: VI to XII. The film is designed to portray and summarize the fundamental concepts of two-figure division after the basic laboratory period has been explored by the pupils. Teachers' guide available. Evaluation: Fair.

11. *Multiplication and Division* (7). A 35-mm. filmstrip, 70 frames in black and white. Grade level: X and up. This filmstrip is suitable for a refresher course in arithmetic. Teachers' guide available. Evaluation: Good.

12. *Square Root and Cube Root* (7). A 35-mm. filmstrip, 52 frames in black and white. Grade level: X and up. This filmstrip is suitable for a refresher course. Teachers' guide available. Evaluation: Fair.

13. *Order of Operations* (7). A 35-mm. filmstrip, 46 frames in black and white. Grade level: X and up. This filmstrip is suitable for a refresher course in arithmetic. Teachers' guide available. Evaluation: Fair.

Fractions

1. *We Discover Fractions* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Harold P. Fawcett. Grade level: IV to VI. This film provides an introduction to the study of fractions and an enrichment of this study. Teachers' guide available. Evaluation: Good.

2. *What Is a Fraction?* (3). A 35-mm. filmstrip in color. Grade level: IV. The filmstrip explains the basic concept of a fraction as an equal part of an object. Teachers' manual available. Evaluation: Excellent.

3. *What Are Fractions?* (4). One reel, 16-mm. sound film; black and white. Consultant: Robert L. Morton. Grade level: IV to VI. The film uses concrete objects familiar to children to introduce basic concepts of fractions. Teachers' guide available. Evaluation: Excellent.

4. *Introduction to Fractions* (8). One reel, 16-mm. sound film; black and white and in color. Grade level: IV to V. This film defines fractions, explains how they

are written, shows the meaning of the terms, and gives several simple problems in determining the value of a fractional part. Teachers' guide available. Evaluation: Excellent.

5. *Parts of Things* (26). One reel, 16-mm. sound film; black and white. Advisers: William A. Brownell and Laura K. Eads. Grade level: II to IV. The film develops the meanings of one-half and one-fourth of single things. Teachers' guide available. Evaluation: Excellent.

6. *Writing Fractions* (3). A 35-mm. filmstrip in color. Grade level: IV to V. The filmstrip explains the significance of writing fractions and stresses the meaning of the terms used in fractions. Teachers' guide available. Evaluation: Excellent.

7. *Simple Fractions* (10). One reel, 16-mm. sound film; black and white. Grade level: V to VIII. The film shows the meaning of fractions, the roles played by the terms of the fraction, and introduces simple additions of fractions. Evaluation: Fair.

8. *Units and Fractional Parts* (15). A 35-mm. filmstrip in black and white. Grade level: V to VI. Suitable for remedial or review work. Evaluation: Fair.

9. *How Large Is a Fraction?* (3). A 35-mm. filmstrip in color. Grade level: IV to V. The filmstrip develops the concept of relative size of such fractions as $\frac{1}{2}$ and $\frac{1}{4}$. Teachers' guide available. Evaluation: Excellent.

10. *Multiple Fractions: Numerator and Denominator* (15). A 35-mm. filmstrip in black and white. Grade level: IV to VI. Suitable for remedial or review work. Evaluation: Fair.

11. *Fractions of a Group* (3). A 35-mm. filmstrip in color. Grade level: IV to V. The filmstrip shows that any group may have fractional parts and then introduces fractions of a group where the equal parts are more than one. Teachers' guide available. Evaluation: Excellent.

12. *How To Add Fractions* (8). One reel, 16-mm. sound film; black and white and in color. Grade level: V to VI. The film presents the addition of fractions having like and unlike denominators. Teachers' guide available. Evaluation: Excellent.

13. *Adding Fractions* (3). A 35-mm. filmstrip in color. Grade level: V to VI. This filmstrip concentrates on the points of greatest difficulty in adding fractions. Teachers' guide available. Evaluation: Excellent.

14. *How To Subtract Fractions* (8). One reel, 16-mm. sound film; black and white and in color. Grade level: V to VI. The film shows how the process of subtraction is applied to fractions and extends the study to include the subtraction of a fraction from a mixed number and a whole number. Teachers' guide available. Evaluation: Excellent.

15. *Comparing Fractions: Adding and Subtracting* (15). A 35-mm. filmstrip in black and white. Grade level: V to VI. Suitable for remedial or review work. Evaluation: Fair.

16. *Addition and Subtraction of Fractions* (7). A 35-mm. filmstrip in black and white. Grade level: X and up. This filmstrip contains four instructional units entitled: "Common Denominators," "Least Common Denominators," "Mixed

Numbers," and "Anticipation of Results." It is suitable for a refresher course in arithmetic. Teachers' guide available. Evaluation: Fair.

17. *How To Change Fractions* (8). One reel, 16-mm. sound film; black and white and in color. Grade level: V to VI. The film is primarily concerned with the reasoning behind the methods of finding equivalent fractions. Teachers' guide available. Evaluation: Excellent.

18. *Common Denominators* (3). A 35-mm. filmstrip in color. Grade level: V to VI. The presentation in this film is restricted to finding a common denominator for $\frac{1}{2}$ and $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{4}$ in problems with addition and subtraction of fractions. Teachers' guide available. Evaluation: Excellent.

19. *Multiple Fractions; Improper Fractions* (15). A 35-mm. filmstrip in black and white. Grade level: V to VI. Suitable for remedial or review work. Evaluation: Fair.

20. *Mixed Numbers* (3). A 35-mm. filmstrip in color. Grade level: V to VI. The filmstrip explains a mixed number and shows how it may be changed to an improper fraction. Teachers' guide available. Evaluation: Excellent.

21. *Using Mixed Numbers* (3). A 35-mm. filmstrip in color. Grade level: V to VI. The filmstrip explains and defines an improper fraction and shows how to change it to a mixed number. Teachers' guide available. Evaluation: Excellent.

22. *Improper Fractions; Mixed Numbers* (15). A 35-mm. filmstrip in black and white. Grade level: V to VI. Suitable for remedial or review work. Evaluation: Fair.

23. *Reducing and Changing Fractions* (15). A 35-mm. filmstrip in black and white. Grade level: V to VI. Suitable for remedial or review work. Evaluation: Fair.

24. *Changing Fractions to a Common Denominator, Part A; Changing Fractions to a Common Denominator, Part B* (15). Two 35-mm. filmstrips in black and white. Grade level: V to VI. Suitable for remedial or review work. Evaluation: Fair.

25. *How To Multiply Fractions* (8). One reel, 16-mm. sound film; black and white and in color. Grade level: VI to VII. This film is definitely for review purposes. It shows what actually happens to a fractional part when it is multiplied by another fraction. Teachers' guide available. Evaluation: Very good.

26. *Multiplying Fractions by Fractions* (3). A 35-mm. filmstrip in color. Grade level: V to VI. This filmstrip gives visual examples of multiplying $\frac{1}{2} \times \frac{1}{2}$, $\frac{1}{2} \times \frac{1}{4}$, and $\frac{1}{2} \times \frac{1}{3}$, with many practice exercises. Teachers' guide available. Evaluation: Very good.

27. *Multiplying Fractions* (10). One reel, 16-mm. sound film; black and white. Grade level: V to IX. The film develops the process of multiplication of fractions through problem situations. It is valuable for review purposes and teacher training. Evaluation: Good.

28. *Multiplying Fractions* (15). A 35-mm. filmstrip in black and white. Grade level: V to VI. Suitable for remedial or review work. Evaluation: Fair.

29. *How To Divide Fractions* (8). One reel, 16-mm. sound film; black and white and in color. Grade level: VI to VII. The film establishes the measurement con-

cept of division and then slowly develops the rule for dividing by a fraction. Teachers' guide available. Evaluation: Good.

30. *Dividing Fractions* (15). A 35-mm. filmstrip in black and white. Grade level: VI to VII. Suitable for remedial or review work. Evaluation: Fair.

31. *Reciprocals: The Rule of Division* (15). A 35-mm. filmstrip in black and white. Grade level: VI to VII. Suitable for remedial or review work. Evaluation: Fair.

32. *Multiplication and Division of Fractions* (7). A 35-mm. filmstrip, 30 frames in black and white. Grade level: X and up. Suitable for a refresher course in arithmetic. Evaluation: Fair.

Decimals

1. *What Are Decimals?* (4) One reel, 16-mm. sound film in black and white Grade level: VI to VII. This film provides an introduction to decimals. Evaluation: Very good.

2. *Decimal Fractions* (8). One reel, 16-mm. sound film; black and white and in color. This film is intended for use in conjunction with classroom instruction and introduces decimals as a special form of common fractions. Teachers' guide available. Evaluation: Very good.

Per Cent

1. *Percentage* (8). One reel, 16-mm. sound film, black and white and in color. Grade level: VII and up. This film presents percentage problems generally classified as Case I or Case II. Teachers' guide available. Evaluation: Good.

2. *Per Cent in Everyday Life* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: H. C. Christofferson. Grade level: VII to X. This film emphasizes the social application of per cent in the solution of problems involving commission, taxes, interest, and discount. Teachers' guide available. Evaluation: Fair.

3. *The Meaning of Percentage* (26). One real, 16-mm. sound film in black and white. Advisers: William A. Brownell and Laura K. Eads. Grade level: VI to VII. The film develops the meaning of per cent in its relation to fractions and decimals. Teachers' guide available. Evaluation: Excellent.

4. *Fractions, Decimals, and Percentage* (7). A 35-mm. filmstrip in black and white. Grade level: X and up. Suitable for a refresher course in arithmetic. Evaluation: Fair.

Measurement

1. *Measurement* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Harold P. Fawcett. Grade level: V to IX. This film presents the importance of measurement in our society and illustrates the use of measures of time, liquids, temperature, weight, length, area, and volume. Teachers' guide available. Evaluation: Very good.

2. *The Story of Measurement* (15). A 35-mm. filmstrip in black and white. Grade level: V to VII. This filmstrip portrays graphically the history and development of measurement and explains the different standards of measure: Evaluation: Fair.

3. *Linear Measure* (15). A 35-mm. filmstrip in black and white. Grade level. V to VII. This filmstrip develops and compares the units of linear measure which include the inch, foot, yard, meter, rod, and mile. Evaluation: Fair.

4. *A Study of Measurement* (15). A series of six 35-mm. filmstrips in black and white. Grade level: VII to IX. These six filmstrips develop the characteristics of a square, rectangle, parallelogram, triangle, trapezoid, and circle. Formulas for measurement of these areas are then developed. Reviews are made frequently. Evaluation: Fair. Titles of these filmstrips are: "Lines and Angles; Surface Measure," "Area of Rectangles; Area of Parallelograms," "Area of Triangles; Area of Trapezoids," "The Circle," "Cubic Measure; Volume," and "Special Measurement: Board Feet, Right Triangle."

Social Applications

1. *The Language of Graphs* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: H. C. Christofferson. Grade level: VII to XII. This film can be used to introduce, to develop, and to review the study of graphs. Teachers' guide available. Evaluation: Very good.

2. *Graph Uses* (7). A 35-mm. filmstrip in black and white. Grade level: X and up. This filmstrip explains the construction and use of a simple bar graph and shows how to make a graph of two variables. Teachers' manual available. Evaluation: Good.

3. *Principles of Scale Drawing* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Harold P. Fawcett. Grade level: VII to XII. This film emphasizes such concepts and techniques as determining scale and the vital importance of scale drawing in modern industry. Teachers' guide available. Evaluation: Excellent.

4. *Scales and Models* (7). A 35-mm. film strip in black and white. Grade level: X and up. This film discusses ratio and proportion and then makes practical applications in the use of scales in ratios, drawings, blueprints, maps, charts, and models. Teachers' manual available. Evaluation: Good.

5. *Maps Are Fun* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Viola Theman. Grade level: V to IX. The purposes of this film are to teach the reading and making of maps and to present the concept that maps are efficient means of communicating certain ideas. Teachers' guide available. Evaluation: Very good.

6. *What Is Money?* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Paul L. Salsgiver. Grade level: V to X. The film shows how money functions as a standard of value, a standard for future payment, a storehouse of value, and a convenient medium of exchange for goods and services. Teachers' guide available. Evaluation: Very good.

7. *Your Thrift Habits* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Paul L. Salsgiver. Grade level: VI to XII. This film aims to provide a general understanding of an effective way to plan savings and spendings. Teachers' guide available. Evaluation: Good.

8. *Using the Bank* (2). One reel, 16-mm. sound film; black and white and in

color. Collaborator: John R. Clark. Grade level: V to IX. This film explains the various services and functions of a bank. Teachers' guide available. Evaluation: Very good.

9. *Fred Meets a Bank* (1). One reel, 16-mm. sound film; black and white and in color. Collaborators: I. O. Foster and F. G. Neel. Grade level: V to X. This film provides an excellent over-all view of banking procedures. Teachers' guide available. Evaluation: Excellent.

10. *What Is Business?* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Paul L. Salsgiver. Grade level: VII to XII. The film attempts to help pupils obtain an understanding of the characteristics of business and an appreciation of the role of business in modern society. Teachers' guide available. Evaluation: Excellent.

11. *Banks and Credit* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: James H. Dodd. Grade level: VIII to XII. The purpose of this film is to show the nature and activities of a commercial bank; to define credit and how it is created, transferred, and put to work by the bank for the needs of the community. Teachers' guide available. Evaluation: Very good.

12. *Installment Buying* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Albert Haring. Grade level: VIII to XII. This film provides an unbiased approach to the topic of installment buying and should be effective as a supplement to classroom work in consumer arithmetic. Teachers' guide available. Evaluation: Good.

13. *Consumer Protection* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Elvin S. Eyster. Grade level: VII to XII. This film is useful to motivate an interest in careful selection and wise buying and to indicate some of the aids that various agencies afford to help one to buy wisely. Teachers' guide available. Evaluation: Very good.

14. *Sharing Economic Risks* (1). One reel, 16-mm. sound film; black and white and in color. Collaborator: Paul L. Salsgiver. Grade level: VIII to XII. This film is suitable for introducing a unit on insurance. Teachers' guide available. Evaluation: Excellent.

15. *Property Taxation* (2). One reel, 16-mm. sound film; black and white and in color. Collaborator: H. F. Alderfer. Grade level: VII to XII. This film explains the necessity for property taxation and shows how assessments are made. Teachers' guide available. Evaluation: Excellent.

16. *Work of the Stock Exchange* (1). One reel, 16-mm. sound film; black and white and in color. Collaborators: John V. Tinen and Sidney L. Parry. Grade level: VIII to XII. This film is suitable for use as a background to precede the study of investments and stocks and bonds. Teachers' guide available. Evaluation: Excellent.

Reference List of Manufacturers or Distributors of Visual and Manipulative Aids

1. Coronet Films, Coronet Building, Chicago 1, Illinois
2. Encyclopedia Britannica Films, Inc., Wilmette, Illinois
3. Eye Gate House, Inc., 330 West 42nd St., New York 18, New York

4. Films Incorporated, 330 West 42nd St., New York 18, New York
5. Ginn & Company, Statler Building, Boston 17, Massachusetts
6. Ideal School Supply Co., 8312 Birkhoff Ave., Chicago 20, Illinois
7. Jam Handy Organization, 2821 East Grand Blvd., Detroit 11, Michigan
8. Johnson Hunt Productions, 1133 North Highland Ave., Hollywood 38, California
9. Kenworthy Educational Service, Buffalo 3, New York
10. Knowledge Builders, 625 Madison Ave., New York 22, New York
11. Little Red School House, Inc., Manasquan, New Jersey
12. Milton Bradley Co., Springfield, Massachusetts
13. Noble & Noble, Publishers, Inc., 67 Irving Place, New York 3, New York
14. F. A. Owen Publishing Co., Dansville, New York
15. Photo & Sound Production, 116 Natoma St., San Francisco 5, California
16. Plymouth Press, 2921 W. Sixty-third St., Chicago 29, Illinois
17. Popular Science Publishing Co., Inc., 353 Fourth Ave., New York 10, New York
18. Scott, Foresman, & Company, 433 E. Erie St., Chicago 11, Illinois
19. Stanley Bowmar Co., 2067 Broadway, New York 23, New York
20. Steck Company, Austin, Texas
21. Superintendent of Documents, Government Printing Office, Washington, D.C.
22. Teach-A-Number Co., 725 Polydras St., New Orleans, Louisiana
23. Teaching Film Custodians, Inc., 25 West Forty-third St., New York 18, New York
24. John C. Winston Co., 1010 Arch St., Philadelphia 7, Pennsylvania
25. Yoder Instruments, East Palestine, Ohio
26. Young America Films, Inc., 18 East Forty-first St., New York 17, New York

CHAPTER X

TESTING INSTRUMENTS AND PRACTICES IN RELATION TO PRESENT CONCEPTS OF TEACHING ARITHMETIC

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OUTSTANDING CHARACTERISTICS OF PRESENT CONCEPTS OF TEACHING

A discussion of testing instruments and practices in relation to present concepts of teaching requires identification of the major characteristics of teaching. Five such characteristics are listed and briefly described.

As has already been indicated in the preceding chapters, present-day teachers of arithmetic are much concerned with developing understanding of the content of the subject. While various ways for attaining understanding of arithmetic are proposed, nearly all place great reliance on teaching the relationships within the number system. The use of social settings within the child's experience is emphasized by many. Another popular procedure emphasizes the use of concrete materials in initial work with both whole numbers and fractions. For example, in teaching the basic ideas of addition, two or more groups of objects are combined into one group and the total number of objects in the group is then determined. Extensive use of pictures and drawings is another characteristic of teaching for understanding. Associated with or appearing as an integral part of these various ways of emphasizing understanding of the arithmetic taught are explanations by text and teacher.

Second, instruction in arithmetic recognizes that the subject involves much more than the ability to compute. Furthermore, most teachers and writers of children's books recognize that not all the children possess a desire to compute with figures. To create interest, many examples of the uses of numbers and the necessity for computing with them are presented. Recognition of the fact that arithmetic involves more than computation has also led to the use of situations in which the pupil is required to make suggestions regarding needed quantitative data or to explain why one way of expressing numerical facts is better than others. The following examples are illustrative. "Harry's rabbit won't stay on the scales. How

can he find the rabbit's weight?" "Which tree is the higher?" The two trees are separated by a tall building but they cast shadows on level ground. "Why is it better to give the height of an underpass in feet and inches instead of just inches?"

The conscious effort made to start instruction at the child's level is a third characteristic of present-day instruction. The use of this guide for teaching has led to acceptance of immature methods of solving quantitative problems such as counting by 1's, 2's, or other groups in beginning simple addition, the use of objects, diagrams and drawings, and the like. The widespread acceptance of "improve your own ability" as a goal instead of "reach the norm for your grade" may be partially attributed to the attempt to start instruction at the child's level rather than the grade level in which the child is classified.

A fourth characteristic of teaching today which seems to be receiving increased emphasis is the attempt to facilitate the use of facts and processes and to give the pupil an opportunity to figure things out instead of relying entirely on telling as a means of instruction. There is also increased emphasis on figuring out the why of procedures.

A fifth characteristic of teaching arithmetic today is the systematic provision for reviews and reteaching. Although some still seem to assume that once taught is sufficient for facts or processes, the majority of teachers and many texts now assume that some forgetting is normal and that simply because such items as the reading of numbers of seven places may be listed for fifth grade in the course of study does not mean that this phase of arithmetic does not need some attention in sixth and seventh grades. Of course, the amount of forgetting and, therefore, the necessity for reviews or reteaching is influenced by the kind and quality of initial instruction.

Current Testing Instruments and Practices

The inclusion of a chapter on testing in this yearbook is some indication of the importance of tests in the arithmetic program. There are scores of commercially published tests in the field of arithmetic. Further evidence of their importance is found in the fact that practically every textbook series offers an extensive testing program, many courses of study in arithmetic have sections devoted to testing or to the broader topic of measurement, and some educational periodicals within the past three years have had an arithmetic test in every issue. In addition, teachers devote a large amount of time to finding out what pupils know.

The nature of arithmetic is such that in the early days of the testing movement test-makers and teachers had no doubts about their ability to construct dependable tests in that field. It would appear that with

such an early start and with the extensive use that is made of tests in arithmetic today, a high level of efficiency in test construction would have been reached. Unfortunately, that is not the case. While arithmetic tests are valuable, a fact supported by their wide use, less progress has been made in the improvement of arithmetic tests and practices than has been made in testing in most other subject-matter areas.

The following description may be of value in presenting a picture of current testing practices.

In the typical middle-grade classroom the teacher has for examination the results of a standardized achievement test in arithmetic given at the close of the preceding term. A school-inventory test is available for administration during the first week of school. In addition, the adopted text has several pages of tests under the heading, "Finding Out What You Remember," and diagnostic tests on some phases of number also appear early in the text. Thus, fully one-fourth of the child's first two weeks of arithmetic is devoted to testing. As instruction continues, the teacher may use regular practice or problem exercises as tests, and, if report card grades are given at the end of four or six weeks, a more formal teacher-made or a teacher-selected test is administered. The results of this formal test are given much weight in the grades assigned. Such tests are usually administered just prior to the close of each report period throughout the year.

If after six to eight weeks of school some or all the pupils seem weak in some phases of arithmetic, a group diagnostic test may be administered, and then some of the students who are experiencing greatest difficulty may be given an individual diagnostic test.

The textbook tests, usually found at the close of chapters, are administered as a regular part of instruction, and the teacher continues to use some problem exercises for testing purposes. At the close of the first semester a standard survey test of achievement or a local school-system test is administered. Then, at the close of the school year, a standardized survey-of-achievement test battery containing an arithmetic section is administered.

In this description of the testing practices of a single classroom, two important purposes of tests are identified: (*a*) determining the status of pupil's mastery of arithmetic as measured by the survey-of-achievement type of test, and (*b*) determining where a pupil's knowledge or skill breaks down, as indicated by the diagnostic test. A third, though much less common purpose of tests, that of determining readiness for new work, is not so easily identified. Some inventory tests have determination of readiness as an objective.

Several sources of tests were identified in the description of classroom testing presented above. Those identified are the teacher-made or teach-

er-selected tests, tests developed for the school system, tests supplied by the textbook, and commercial or published tests in the form of survey and diagnostic tests. For convenience in discussion, the teacher-made and school-system tests will be put in one category and the other two, textbook and published tests, will be treated separately. Of course, these names are only loosely applied for some standardized and diagnostic tests are teacher-made, some school-system tests have standards, and there are published teacher-made tests. Under the heading, teacher-made tests, will be considered all tests made by teachers, supervisors or others directly connected with the school system and also the suggested tests which frequently appear in such magazines as the *Grade Teacher*. The majority of teacher-made tests are similar in form and content to the subject matter that is presented in the class periods just prior to the giving of the tests. "Find the sums for these examples." "Answer these problems," and the like, are representative of the directions used. This concern of teacher-made tests with that which has only recently been studied has some advantages, such as, avoiding the use of valuable pupil time on phases of the subject not yet studied. Usually only an exact answer to the computation involved is the acceptable answer on most teacher-made tests. To permit pupils to find out whether or not they can do the things just studied has merit and is unquestionably an important motivating factor in many classrooms. By careful scrutiny of pupil's test papers, teachers are frequently able to diagnose a pupil's weaknesses and can use that knowledge as a guide in further instruction.

The fact that teacher-made tests are concerned primarily with limited aspects of arithmetic such as exact computational procedures is a major shortcoming. It has already been noted that exact answers to suggested computational exercises are the only ones acceptable. While the importance of exact computation is recognized, the fact should be noted that in the majority of life situations where quantity is involved approximate answers are sufficient. Another weakness of these tests is the failure to include items which deal with study procedures pupils employ in acquiring arithmetic skills and understanding. After all, the pupil spends a considerable amount of his arithmetic study time in acquiring skill and knowledge. The use of form and content similar to what has just been studied, while having merit, has from the standpoint of testing very obvious weaknesses. A pupil giving the correct response may only be remembering the form of the item studied a few days prior to test time.

Textbook Tests. The tests found in textbooks and those supplied to teachers in the manual, or as part of supplementary materials, are in general superior to teacher-made tests. Many of them are designed to survey achievement, to determine readiness, or to be used for diagnostic pur-

poses. It is doubtful that the tests actually meet these different purposes as fully as the claims made for them would imply. Nevertheless, recognition of a need for different tests to meet different purposes is an advance over the single test once used for all purposes. The diagnostic tests included in textbooks are usually accompanied by page reference to the section of the book where the subject matter of the items is first presented. Such practices seem desirable and in keeping with the principle of teaching which states that instruction should begin at the child's level.

Although rather rare, some textbooks have begun to include in tests items other than computational exercises and problems. Such items as oral tests, what's wrong here, tests without numbers, and the like, are introduced.

The shortcomings listed for teacher-made tests are, in the main, applicable to textbook tests. Consider, for example, the point concerning use of only the one exact numerical answer. The extent of this emphasis on exact computation is shown by the fact that every test item of the first thirty in two recently published texts for the fifth grade calls for an exact answer. The child who makes only a minor error gets no more credit than the child who has no idea about the solution of the problem.

The use of goals or standards on textbook tests has both advantages and disadvantages. The provision of some sort of a goal is generally considered an advantage in learning. However, the goals or standards on textbook tests, if not easily attainable for poor students, become a source of discouragement to these students. If goals are easily attained by the poorer students, then there is little likelihood that such goals will be of much value to the average or the superior student.

Published Tests (Standardized). Most standard tests in arithmetic are of the general survey-of-achievement type. As indicated by the title, these tests have standards or norms. A given score can be changed to a grade equivalent, percentile rank, or some other standard measure which permits comparisons with the status of pupils in other schools. A more important use of standards is the means which they afford of determining individual pupil growth. To know that a pupil has grown .8 of a grade in a year is far more satisfying and meaningful to pupil, teacher, and parent than are grades of "B" or of "C" on the end-of-year arithmetic tests. Without standard tests, progress is difficult to gauge, and, without standards, general surveys of achievement do not mean much.

The commercial standardized tests, like the other two categories discussed, are limited in coverage. Very few of them even attempt to measure in such a common area of the subject as oral arithmetic. Most general survey-of-achievement tests sample so briefly from the many areas of arithmetic that they have little value for diagnostic purposes. The stand-

ards for the tests are far from being as reliable as is essential to the purposes for which they are used. Only a few of the most reliable yield measures that are stable enough to permit the graphing of growth over a period of years. Another shortcoming of norms on most standard tests is due to the use of so-called national norms—norms obtained by using the performance of pupils from various sections of the country and from many kinds of schools. Certainly these population groups are difficult to describe, and yet a pupil's grade equivalent is given in terms of the average attainment of such a population. It would seem that a much better description of an individual's achievement might be made if norms were based on the status of a more homogeneous population, one more easily described. Many teachers and supervisors in large cities wish that they might have large-city norms. In a like manner, teachers and supervisors in rural areas would like to describe the performance of their pupils in terms of rural populations.

How tests are to be scored is an item of major concern in all tests. Even in a field as objective as is arithmetic, marked variations in assigned scores are obtained when a number of teachers grade the same papers. Most standardized tests are accompanied by keys, but even so, comparison between the answer on the key and the pupils' papers has to be made. Such comparisons require time and a great deal of energy when pupils' answers resemble but are not exactly the same as those on the key. Some tests are of the recognition type which, when used in connection with a stencil or peep-hole-type scoring key, markedly reduce the time and energy required for scoring tests. The use of such time-saving features is highly desirable.

To determine where a pupil's knowledge breaks down was listed as one of the major purposes of tests. This information serves the function of diagnosis. The most widely used practice of teachers in diagnosis is the observation of the pupil's oral and written work. This procedure is such an integral part of instruction that many teachers do not even think of the practice as testing. It is, however, a very important testing procedure and should be encouraged. Closely associated with the observation technique is the interview with the pupil regarding his daily work or his solution or attempted solution of items of a test. The interview, like the observation procedures mentioned above, is a desirable feature of the guidance program in the school.

The commercial diagnostic tests are widely variable in character. Some are very comprehensive, while others are relatively brief. Some are designed to be administered to individual pupils, while others are designed for group administration. The Buswell-John Diagnostic Chart for Fundamental Processes illustrates the individual type. It consists of a chart

for the administrator and a test sheet for the pupil who is directed to do the work aloud. Obviously, such tests cannot be given to very many pupils. The Compass Diagnostic Tests which are for group administration consist of twenty tests, which is again a formidable undertaking when large numbers of pupils are to be tested. Of course, the whole series of tests would not be administered, but even the use of several of the tests would require a great deal of time for the testing and scoring. Those diagnostic tests which are easily administered sample different areas of the subject so briefly that the reliability of the test is, of necessity, low. From study of the reviews of commercial diagnostic tests in the several issues of the *Mental Measurements Yearbook* one can only say, "Diagnostic tests either must be markedly improved or we must conclude that such tests can only be recommended for specialists such as clinicians or remedial teachers."

Recently committees of teachers and principals of large city systems have attacked the problem of diagnosis of pupils' difficulties.[1] Their reports offer some valuable suggestions and should be influential in promoting interest and work in this field of testing. Among the many testing procedures suggested for diagnosing weaknesses, these two are noteworthy: "supplying needed facts to solve the problem," (e.g., number of days in a week) and "the use of a test on problems without numbers." The latter procedure indicates a lack of confidence in the usual problem item found in tests. Other city systems have recognized the necessity for preparing diagnostic tests for teacher use. Such efforts should be of much assistance to teachers.

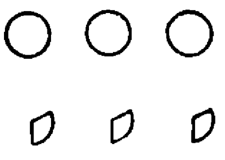
Commercially distributed readiness tests in arithmetic have not reached any high level of popularity, and, as a result, only a few are on the market. Little is known concerning the uses of such tests in beginning arithmetic, and, even if low scores are revealed by the tests, we know little about the program needed for pupils who make low scores. Furthermore, resourceful teachers are already using the best-known programs for building number background without reference to the findings of readiness tests. Readiness tests in upper-grade arithmetic have been even less popular than those for pupils in the primary grades. Whether this lack of progress in readiness testing in arithmetic is due to teacher indifference, to the great emphasis given to reading readiness, to the nature of arithmetic, to the lack of satisfactory testing instruments, or to some other reason is not known. That there is little interest at present is probably the best statement that can be made on the status of arithmetic-readiness testing.

The tests and practices mentioned in the preceding sections of this

[1] "Arithmetic Teaching Techniques," *Chicago Schools Journal*, XXX (March, 1949), 211–12.

chapter have all been of the pencil-and-paper type. There are other methods of testing which have recently received a great deal of attention, some of which give promise of being of value in the total program. For want of a better name, "non-pencil-and-paper tests" will be used to identify them. As has already been indicated, a test is a means of determining how much mastery a pupil has of the material under consideration. One of the best indications of the mastery of a subject possessed by a pupil is his ability to make significant comments or to ask intelligent questions about the subject. As a pupil works in arithmetic classes, the teachers have many opportunities to observe the above and thereby obtain an idea of the pupil's grasp of various phases of arithmetic. Consider, for example, a pupil's work and explanation for the problem, "At $1\frac{1}{3}$¢ a pound how many pounds of scrap iron will Bill have to sell to earn a dollar?"

3 lb.



4¢

6 lb.	18 lb.	54 lb.	72¢	54 lb.
8¢	24¢	72¢	24¢	18 lb.
			96¢	72 lb.
			4¢	3 lb.
			$1.00	75 lb.

"First I used drawings and found that 3 lbs. would bring 4¢. I doubled that and then multiplied by three until I thought I had a dollar's worth."

Another indication of achievement in a field is interest in that field, for few human beings maintain interest for any length of time in a field where they possess little knowledge. Still another indication of achievement is the degree of confidence displayed when work is assigned or undertaken. Other situations which permit teachers to see whether or not the pupil has achieved success are these: (*a*) his use of numbers in other than the more or less artificial arithmetic situations; (*b*) his ingenuity when confronted with a new number situation; (*c*) his thinking or rationalizing when trying to decide whether or not an obtained answer is reasonable.

Illustrations for these three situations follow.

(*a*) In giving a report on the New England States a pupil said, "You can see from the map that Maine is about four times as large as Massachusetts. I multiplied 8,257, the area of Massachusetts, by 4 and got 33,028 sq. mi., which is a little less than the 33,215 sq. mi. which is the area of Maine."

(*b*) A pupil who had not had instruction in multiplication with a two-figure multiplier, when confronted with the problem 14 × 68, offered the following:

$$\begin{array}{r} 68 \\ \times 7 \\ \hline 476 \\ 476 \\ \hline 952 \end{array}$$

(*c*) In judging whether or not 3,198 was a reasonable product for $9\frac{3}{4} \times 328$, a pupil said, "If $9\frac{3}{4}$ had been 10 the product would have been 3,280. 3,198 is close to that."

Evidence obtained through such situations as those described above, combined with evidence from the usual pencil-and-paper tests is much more likely to give a true picture of a pupil's achievement than is that based solely on the usual pencil-and-paper tests. In securing evidence of understanding these non-pencil-and-paper situations are superior. The nature of such testing practices makes standardization undesirable and also precludes their use in formal testing periods. The difficulty experienced by teachers in keeping a record of evidence obtained through non-pencil-and-paper test situations is one of the major weaknesses of this type of testing. The keeping of individual arithmetic folders into which notes on pupil performance, actual samples of work, and the like, are placed seems to be the most satisfactory method of assembling such data. The use of non-pencil-and-paper tests not only affords better measures of pupil achievement than is obtained through sole use of the usual pencil-and-paper tests but also encourages the use of new and promising teaching procedures. On the other hand, the pencil-and-paper tests tend to maintain the *status quo* and thus become a block to progress. Efforts to improve the non-pencil-and-paper types of test should, therefore, be encouraged in every possible way.

A discussion of current tests and testing practices would be incomplete without mention of the value of tests as learning exercises. That tests are motivating factors has already been mentioned. The learning that occurs in the taking of a text and in the discovery and correction of errors revealed by the test are not generally recognized, but that such learning is considerable in amount is the almost unanimous opinion of those who have carefully studied teaching procedures. Appropriate recognition of the learning outcomes associated with a liberal use of test exercises affords substantial justification of the large portion of arithmetic time devoted to testing.

From this discussion of tests and testing practices it can be seen that, while tests are of recognized value and are extensively used, great changes have to take place before a really satisfactory testing program is achieved.

Arithmetic Tests in Recent Publications

Recent books on the teaching of arithmetic continue to give to tests and test practices an important place. There are found, however, frequent statements and suggestions which call attention to the inadequacy of present tests. The following is representative: "There are no standard

tests for many of the outcomes listed. . . ."[2] While a number of suggestions are made in the textbooks to meet the apparent deficiencies of current test practices, no radical departure from past practices has resulted from these suggestions. A few magazine articles have made some specific recommendations regarding tests and general measurement procedures. Sueltz[3] suggests a number of novel test exercises. The following are representative: "How many inches long is the pencil drawn below?" "The meter shows the number of gallons of water used. When another gallon is used what number will the meter show?"

The importance of tests in the instructional procedures used by better teachers is emphasized in a recent article.[4] Attention is called in this article to the desirability of having separate tests to correspond to important objectives and of measuring the *why* and *how* of operations.

Recent books on tests and measurements, like the books on the teaching of arithmetic, give testing in arithmetic an important place but, unlike the latter, give little in the way of suggestions for improvement or change. Practically all the space in such books is devoted to conventional-type pencil-and-paper tests.

Yearbooks of professional organizations, notably the Sixteenth Yearbook of the National Council of Teachers of Mathematics, and Part I of the Forty-fifth Yearbook of the National Society for the Study of Education, contain valuable discussions of and suggestions for improvement in the teaching of arithmetic. In chapter x of the Sixteenth Yearbook of the National Council, Brownell calls attention to the inconsistency of trying to separate testing practices from instructional practices. In his list of suggestions for testing are included such important items as (*a*) specific limitations of usual tests, (*b*) importance of teacher observations, (*c*) use of individual interviews and conferences, (*d*) value of pupil reports, projects, and the like, and (*e*) a list of promising test devices.

The current emphasis on understanding in the teaching of arithmetic makes the *Measurement of Understanding,* Part I of the Forty-fifth Yearbook of the National Society for the Study of Education, of special significance. Among the many important points of interest in that volume, the following pronouncements are made: (*a*) Considering understandings as educational outcomes is dangerous, for teaching for understanding without reference to use in life may be nonfunctional. (*b*) Tests

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[2] L. J. Brueckner and Foster E. Grossnickle, *How To Make Arithmetic Meaningful,* p. 378. Philadelphia: John C. Winston Co., 1947.

[3] Ben A. Sueltz, "Measurement of Understandings and Judgments in Elementary-School Mathematics," *Mathematics Teacher,* XL (October, 1947), 279–84.

[4] Maurice Hartung, "A Forward Look at Evaluation," *Mathematics Teacher,* XLII (January, 1949), 29–33.

have a directing (sometimes harmful) influence on instruction. (*c*) The day-by-day observations of alert teachers provide the most significant evidence of pupils' understanding. Chapter vii, which is devoted to the measurement of understanding in elementary-school mathematics, contains such recommendations and items as, "Given mathematical conditions need not call for a unique solution," "Why does the multiplication check for division work?" "For a test of understanding of principles and procedures, observation, discussion, and interview are better than pencil-and-paper tests." In the sample test items offered for measurement of mathematical understandings there are but few items of the type found in conventional tests.

The reviews of arithmetic tests in the issues of the *Mental Measurements Yearbook* afford a valuable means of getting a general idea of tests and possible uses for them. The comments, when critical, are usually substantiated. On the basis of these reviews one would conclude that while arithmetic tests are of some value, much improvement in test construction is still to be desired.

Further indication of the inadequacy of current tests and the imperfections of testing practices may be found in late courses of study. For example, the Philadelphia Course of Study, in the section on "Appraising the Child's Growth," suggests the use of check questions in appraising pupils' *attitudes and habits, social understandings and social skills.* The use of pencil-and-paper tests is suggested for those areas where such instruments are applicable. A noteworthy recommendation is the suggested "Record Chart"[5] for each pupil.

A statement from the Washington, D.C., Course of Study expresses the recognized need for specific tests. "Give diagnostic tests to determine children's reasoning and computational abilities in: (1) Understanding fractions, (2) Expressing fractions pictorially, in figures, and in words."[6] A reference to the "Diagnostic tests from the research department" indicates that this school system is attempting to meet the specific need.

Recommendations

The preceding sections of this chapter have called attention to the inadequate nature of current tests and testing practices and have specifically identified some procedures as valuable. In this section definite recommendations concerning tests and practices are made. Before considering such recommendations, this section will summarize briefly some of

[5] *Arithmetic*, pp. 130–33. Philadelphia: Philadelphia Public Schools, 1946.

[6] *Mathematics: Course of Study for Elementary School—Kindergarten through Sixth Grade.* Washington: Public Schools, 1948.

the major weaknesses that have been listed and call attention to others that exist.

The general assumption that all tests are of the pencil-and-paper type might well be considered the number-one weakness of current practices. As a result of this assumption other testing procedures are neglected, and undue weight is placed on results of the pencil-and-paper test. The majority of pencil-and-paper tests are concerned with only limited aspects of arithmetic and emphasize computation rather than understanding in those limited areas. The scoring of many tests is unnecessarily time-consuming, and norms leave much to be desired. The practice of allowing no credit unless the final correct answer is given is an inefficient way of measuring some phases of arithmetic. Good forms and procedures for recording evidence obtained in the non-pencil-and-paper test situations are lacking, with the result that much of the teacher's time and effort are expended on them. Systematic plans for obtaining evidence through non-pencil-and-paper situations have not been generally adopted.

Some recommendations for improvement of tests and testing practices are implied in the weaknesses listed above. However, for the sake of emphasis these implied recommendations and others are given in the statements that follow.

1. It should be recognized that evidence of pupil achievement must be gathered in many ways. The values of observation during everyday work and during planned test situations, pencil-and-paper tests, interviews, use made of numbers in nonarithmetic situations, and so on, must be recognized. Until such recognition is made, maximum progress in testing practices can hardly be expected.
2. To avoid lapses and haphazard procedures, a systematic, planned program of non-pencil-and-paper testing is recommended. Possible items for such a program are listed below.
   a) Situations which call for the gathering of data (numbers) and the use of such data in making judgments, for example, a report on the cost and time of going on an excursion by regular bus, by chartered bus, or by rail; the presentation of two types of picnic lunches for a class picnic with estimates of the amount of food for each person and the cost of that food per person.
   b) Interpretation of quantitative statements from other school subjects or current news. To illustrate, consider these two interpretations for the news headline, "Crowd of 18,000 greets the Governor." One pupil said, "A lot of people." Another, "The crowd was equal to about two-thirds of the people in the county."
   c) The collection of samples of everyday work. Such samples should give the date on which the work was done, show whether they are representative of superior, average, or inferior work for the group, and provide

other information which may be of value in a study of the exhibit. A minimum of three such samples for a year is recommended.

d) The writing of an informal report on each pupil's work in arithmetic at least once each semester. The following is illustrative: "During this semester Frank's attitude toward arithmetic remained rather passive. He seldom showed any enthusiasm even when his work was well done. At no time did he volunteer suggestions regarding the solution of problems. His arithmetic papers are neat."

e) Recording of any special points such as unique solutions, special difficulties, and so on. The illustrations on pages 193–94 are good examples.

f) Keeping an arithmetic folder for each child.

g) Having pupils explain their own solutions to problems or the solutions of other pupils. This means of obtaining evidence is well shown in this situation. "Here is Bob's check for 23 × 47 = 1081. Can you figure out how he thought?

```
   47
  ×11
  ---
   47
  47
  ---
  517
  517
+  47
 ----
 1081
```

The pupil's work and explanation on page 193 is another illustration of this procedure.

3. In order that teachers and children may recognize tests as learning procedures, there should be frank discussion of such questions as, "Why do we take tests?" Simple experiments designed to show that pupils learn through the taking of a test may also be used. For example, a test on such a subject as the multiplication facts may be prepared and divided into two parts, one part of the test to be taken and corrected by the pupils immediately. Later, the entire test is given to the same pupils and the scores on the two parts recorded separately. Unless achievement is at a very high level, the part of the test which was repeated will show the higher score.

4. It is recommended that the areas measured by pencil-and-paper tests be markedly broadened. Among the many areas of arithmetic now rarely included in such pencil-and-paper tests, the following are especially recommended: Oral or mental arithmetic, knowledge of the how and why of processes, forming judgments on the basis of quantitative data, and the selection of one part of a process or the next step in a process. Sample items for each of the above follow.

a) Without doing any figuring, select the best answer to each of these items:
1. 26 + 32 + 98 + 15 = (1) 160. (2) 171. (3) N.

b) In solving 46)$\overline{2358}$, which of these is the first step to be taken? (1) Multiply 46 by 5. (2) Select the first partial dividend. (3) Change the 2358 to 2300.

c) In the problem, 38, why is the 6 placed under the 5?

$$\begin{array}{r} 38 \\ \times 24 \\ \hline 152 \\ 6 \end{array}$$

(1) Because you always skip one to the left.
(2) Because 2× 3 = 6.
(3) Because the 6 is actually 6 tens.

d) The amount of fish caught in 1948 in Lake Mars was 82,852 lb. In 1947 it was 51,036 lb. For a general comparison, what fraction would you use in telling what part the amount caught in 1947 was of the amount caught in 1948? ( ) $\frac{5}{8}$. (2) $\frac{1}{2}$. (3) $\frac{51{,}036}{82{,}852}$.

e) What number operation would be used in solving this problem? "At 28¢ a pound, what will $6\frac{1}{2}$ pounds of ground meat cost?" (1) Addition. (2) Multiplication. (3) Division.

f) In $28\overline{)5367}$ with 1 written above, what is the next step to be taken? (1) Subtract 28 from 53. (2) Bring down the 6. (3) Select the next quotient figure.

5. The sample items in the preceding recommendation are of the recognition type. Because of the ease of scoring, such items are highly recommended for the major parts of all published tests whether they be standardized or just city or district survey tests. The effort teachers expend in scoring tests may be considered as taken from instruction. Therefore, appropriate measures which save time and energy in the scoring of tests should be encouraged. And, as has already been pointed out in an earlier section, the testing for specific aspects of arithmetic, as for example, the identification of the next step in a solution, makes for greater testing efficiency. Furthermore, tests concerned with specific aspects of the subject are needed for diagnosis.

6. Another recommendation for the improvement of conventional tests is the inclusion of items which require only approximate answers. Since by far the greater portion of life situations involving quantity require only approximate answers, it seems reasonable to assume that arithmetic tests should emphasize that type. Consider, for example, the following situation. "In the election, 796 of the 2,022 qualified voters cast ballots." Which of these is the better way of using the quantitative facts? About 800 of 2,000 voted, or 796 of 2,022 voted. Two other items based on the above fact are: (*a*) Only $\frac{4}{10}$ of the voters cast ballots, or $\frac{796}{2022}$ cast ballots; and (*b*) About 1,200 voters did not vote, or 1,226 voters did not vote.

7. The use of language or form which varies from that used in instruction is still another recommendation for the improvement of conventional tests. In division, for example, instead of always using the form $37\overline{)891}$ which is so common in textbook and practice exercises, some of the following might be used: "How many 37's in 891?" "891 ÷ 37 equals what number?" or, "891 is how many times as large as 37?"

8. There is a place for limited use of essay-type questions in the field of arithmetic. For example, in the explanation of procedures, in telling why a check is correct, and the like, the essay-type response gives superior evidence. The

emphasis given to multiple-response questions in an earlier recommendation does not mean that the free-response type of item is to be excluded.

9. The preparation of tests with norms based on selected populations rather than the nation are probably more valuable than the so-called national norms. It is, therefore, recommended that the preparation of special norms such as mid-western cities of population of 50,000 to 150,000 be encouraged. Of much greater importance is the recommendation that norms be used primarily in showing individual pupil growth from year to year. To be highly successful with such growth graphs, norms that are much more reliable than those now provided must be developed.
10. As a concluding or general recommendation, it is suggested that the amount of instructional time of both pupil and teacher be kept at a reasonable figure. Although there are not data to substantiate the figures, from 15 to 20 per cent is suggested. Although tests are a part of instructional procedure, by far the major portion of arithmetic time should be given to other instructional procedures.

A List of Arithmetic Tests

The following list of tests is presented with some misgivings. In the first place, such a list tends to emphasize the type of test which is adaptable to publication and quite obviously takes emphasis away from other means of obtaining evidence of pupil progress. Since the importance of other ways of testing has been emphasized in the chapter, and because a list of published tests is often helpful, such a list has been selected. A second reason for hesitating about listing tests is the impossibility, because of space limitations, of including all tests and the dangers which accompany the selection of any limited number of tests. It was decided to limit the list to those tests published in the United States and reviewed in the 1940 and 1949 editions of the *Mental Measurements Yearbook*. It is suggested that users of this list of tests consult the reviews in these yearbooks.

1. *Analytical Scales of Attainment in Arithmetic*, Grades 3–4, 5–6, 7–8. L. J. Brueckner, Martha Kellog, and M. J. Van Wagener. Minneapolis: Educational Test Bureau, Inc., 1933.

2. *National Achievement Tests*, Grades 3–8. Robert K. Speer and Samuel Smith. Rockville Center, New York: Acorn Publishing Co.

3. *Iowa Every Pupil Tests of Basic Skills*, Test D, Grades 3–5, 6–8. H. F. Spitzer and Others. Boston: Houghton Mifflin Co., 1940.

4. *Compass Diagnostic Tests in Arithmetic*, Grades 2–8. G. M. Ruch, F. B. Knight, H. A. Greene, and J. W. Studebaker. Chicago: Scott Foresman Co.

5. *Compass Survey Tests in Arithmetic*, Grades 2–4, 4–8, *ibid.*

6. *Diagnostic Tests for Fundamental Processes in Arithmetic*, Grades 2–8. G. T. Buswell and Lenora John. Bloomington, Illinois: Public School Publishing Co., 1925.

7. *Kansas Arithmetic Test*, Grades 3–5, 6–8, 1934. Ruth Otterstrom and H. E. Schrammel. Emporia, Kansas: Bureau of Mental Measurements.

8. *Kansas Primary Arithmetic Test*, Grades 1–3, *ibid.*

9. *Sangren-Reidy Survey Tests in Arithmetic*, Grades 2–3, 4–6, 7–9. Paul V. Sangren and Ann Reidy. Bloomington, Illinois: Public School Publishing Co., 1933.

10. *Schorling-Clark-Potter Arithmetic Test*, Grades 5–12. Yonkers-on-Hudson, New York: World Book Co., 1926–28.

11. *Wisconsin Inventory Tests in Arithmetic*, Grades 2–8. W. J. Osburn. Bloomington, Illinois: Public School Publishing Co., 1924–29.

12. *Arithmetic Reasoning*, Grades 9 and over; 1941; 1 form. B. V. Moore. Distributed by Psychological Corporation.

13. *Arithmetic Test* (Fundamentals and Reasoning), Grades 3–6, 6–8 (same as arithmetic tests in *Municipal Battery;* Forms A, B; 2 levels). Robert K. Speer and Samuel Smith. Rockville Center, New York: Acorn Publishing Co., 1938–39.

14. *Basic Arithmetic Skills; Iowa Every-Pupil Tests of Basic Skills.* Test D, New Edition, Grades 3–5, 5–9. IBM, Form O, Grades 5–9. Forms L, M, N, O, 2 levels. H. F. Spitzer in collaboration with Ernest Horn, Maude McBroom, H. A. Greene, and E. F. Lindquist (General Editor). Boston: Houghton Mifflin Co., 1940–45.

15. *Basic Skills in Arithmetic Test*, Grades 6–12; Forms A, B. William L. Wrinkle, Juanita Sanders, Elizabeth H. Kendel. Chicago: Science Research Associates, 1945.

16. *Chicago Arithmetic Readiness Test*, Grades 1.5–2; 1 form. J. T. Johnson. Eau Claire, Wisconsin: E. M. Hale & Co., 1939.

17. *Chicago Arithmetic Survey Tests*, Grades 3–4, 5–6, 7–8. Forms 1R, 2R, 3; 3 levels. J. T. Johnson. Eau Claire, Wisconsin: E. M. Hale & Co., 1939–43.

18. *Clapp-Young Arithmetic Test: Clapp-Young Self-Marking Tests*, Grades 5–8; Forms A, B. Frank L. Clapp and Robert V. Young. New York: Houghton Mifflin Co., 1927–30.

19. *Hundred-Problem Arithmetic Test*, Grades 7–12 (revision of *Schorling-Clark-Potter Arithmetic Test*, Forms V, W. Raleigh Schorling, John R. Clark, and Mary A. Potter. Yonkers, New York: World Book Co., 1926–44.

20. *Lee-Clark Arithmetic Fundamentals Survey Test: High School Edition*, Grades 8–12; Forms A, B.; J. Murray Lee and Willis W. Clark. Los Angeles: California Test Bureau, 1944.

21. *Metropolitan Achievement Tests*(*Arithmetic*) Grades 3–4, 5–6, 7–9.5; Forms R, S. Richard D. Allen, Harold H. Bixler, William L. Conner, and Frederick B. Graham. Yonkers, New York: World Book Co., 1933–47.

22. *Renfrow Survey Tests of Mathematical Skills and Concepts*, Grades 3–4, 5–6. 7–8; Forms A, B. Omer W. Renfrow. Cincinnati: C. A. Gregory Co., 1941.

23. *Schrammel-Otterstrom Arithmetic Test*, Grades 4–6, 7–8. Forms A, B; 2 levels. H. E. Schrammel and Ruth E. Otterstrom. Emporia, Kansas: Bureau of Educational Measurements, Kansas State Teachers College, 1945.

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CHAPTER XI

THE TRAINING OF TEACHERS OF ARITHMETIC

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Three things which determine the nature of instruction in arithmetic are: (*a*) the objectives of the instruction; (*b*) the materials and methods of instruction; and (*c*) the teachers. This chapter deals with the academic and professional preparation of teachers of arithmetic. The data presented in the following pages were taken from the literature dealing with the subject, from a questionnaire sent to state teachers' colleges, and from a study of college catalogues.

Requirements for Certification

Considerable progress has been made in increasing the amount of training for teachers in the elementary school during the past twenty years. At the time of the publication of the National Society's last yearbook on arithmetic, Buckingham (**3**:320) concluded that the two-year program was the prevailing type of training course offered. The four-year course now is the prevailing type of program for the training of elementary teachers.

TABLE 1

Minimum Requirements in Years of Training for Certification of Elementary Teachers in Different States
[Adapted from Woellner and Wood (**16**)]

Years of Professional Study	No. of States
One.	5
Two	18
Three	4
Four	20
No state specifications.	1

Table 1 indicates that twenty states require a minimum of four years of professional study for the certification of elementary teachers. Unfortunately, in five states, one year of study is acceptable for certification.

In all these states a certificate granted at the completion of one year of study beyond high school is for teaching in the rural schools of that state. Except for a limited number of teachers in a few states, two years of professional training is the minimum for certification.

The prescribed standards for certification in most states are somewhat fictitious. Since the close of World War II it has been impossible to secure enough adequately trained teachers for the elementary grades. As a result, in 1948–49 about 300,000 teachers—almost one out of three—held provisional certificates which were substandard. This situation is likely to persist for many years unless more recruits are secured for training for the elementary school.

The serious shortage of teachers in the elementary field can be seen from the estimates made in 1948 by the National Commission of Teacher Education and Professional Standards for teachers at this level. The estimate of the needs is as follows:

> During the ten years beginning in 1949–50 the number of new elementary-school teachers needed will be 1,033,994. Of this number 262,100 will be needed to replace teachers who will quit teaching, 70,000 to replace teachers now on emergency certificates, and 142,460 to reduce the average class size in elementary schools to twenty-five pupils per teacher. The number of elementary-school positions will increase from 643,500 in 1947 to 921,000 in 1957–58, the peak year for elementary-school enrolment—an increase of 277,500 positions. Should the 1947–48 birth rate and the current rate of production of new teachers continue, an estimated shortage of 621,984 elementary-school teachers for new positions and for replacements alone would develop by 1958. The accumulated shortage would exceed 800,000 by 1958. The estimated annual demand for elementary-school teachers during the decade will be 103,399, which is about five times the present level (**1**:182).

As long as the teaching profession faces a shortage of teachers as acute as that described above, instruction in arithmetic—and in other subjects also—cannot be generally maintained at a high professional level.

Some states have certain subject-matter requirements for certification in the elementary school. Table 2 shows the requirements in mathematics, science, and social studies and history. There are other subject-matter areas specified in legislation, but they are not given in the table. The requirements may be either specific or rather general in their application to areas of study. A specific requirement indicates the number of semester hours of credit in a given subject. The more general or "blanket" type of requirement indicates that a certain number of semester hours must be earned within a group of subject-matter fields.

Table 2 shows that ten states require an average of 3.4 semester hours of mathematics as compared with approximately twice as many states

requiring an average of 9.0 semester hours of social studies and history. The table also shows that a specified amount of training is prescribed more frequently in science than in mathematics.

The writer is not urging that legislation should be enacted requiring that more mathematics be specified for certification. The tendency to enact such legislation for particular subject-matter areas is not to be commended. We are here concerned with the requirements for certification of teachers in the elementary school.

TABLE 2*

SPECIFICATIONS IN CERTAIN SUBJECT-MATTER FIELDS FOR CERTIFICATION OF ELEMENTARY TEACHERS IN DIFFERENT STATES

Subject Field	Number of States Having Blanket or No Specific Requirements		Number of States Having Specific Requirements	Number of Semester Hours in Specific Requirements		
	Blanket Requirements	No Requirements		Minimum	Maximum	Average
Mathematics.....	3	35	10	2	6	3.4
Science.....	3	28	17	3	18	8.8
Social studies and history	3	26	19	3	18	9.0

*Adapted from Woellner and Wood (16)

QUESTIONNAIRE STUDY OF TRAINING OF TEACHERS OF ARITHMETIC IN TEACHERS' COLLEGES

The writer sent a three-page questionnaire to 147 state teachers' colleges which are members of the American Association of Colleges for Teacher Education. One hundred thirty-one returns were received, 129 of which were used in this study. Two of the replies could not be used because they came from colleges which offered a specialized program of training. The 131 replies from 147 questionnaires sent out represent a return of about 90 per cent. This is a highly satisfactory response for a three-page questionnaire. Shannon[1] found that the mean number of returns in highly selected questionnaire studies is about 70 per cent.

There are probably two reasons why the returns were so favorable. First, all of the data asked for were objective and factual. Besides, they had a direct bearing on the training of teachers of arithmetic. Second, the writer made an attempt to secure the data from published sources. An analysis was made of each college catalogue and much of the information

[1] J. R. Shannon, "Percentages of Returns of Questionnaires in Reputable Educational Research," *Journal of Educational Research*, XLII (October, 1948), 140.

required for this study was recorded on the inquiry forms before they were sent out. The recipient of the questionnaire was asked to certify the data.

Differentiated Curriculums

It seems logical that the curriculum would be differentiated for the training of teachers in our present age of specialization in scientific fields. There is such a wealth of professional material bearing on the teaching of arithmetic at various grade levels that a teacher who is training for all grades cannot become familiar with many of the scientific studies unless he takes several courses in the teaching of the subject. The teacher who specializes in the work of the primary grades should pursue professional courses in arithmetic which are different from those selected by the teacher who prepares for a position in the intermediate or the advanced grades. The course of study underlying training for the kindergarten-primary grades may extend through the first three grades. The intermediate curriculum is intended for Grades IV, V, and VI. The upper or the advanced curriculum is for Grades VII and VIII. Where provision is made for a general elementary curriculum, it is designed to cover the entire program in the elementary school.

Over twenty years ago Buckingham expressed the view that a general curriculum could be defended only for the preparation of teachers for the rural schools. "Here a virtue must be made of necessity. Since the teacher must teach all the arithmetic, the institution must do the best it can to prepare him for his total job" (**3**:322). If this statement was true more than two decades ago, it must be more pertinent today because of the accumulation of additional scientific material pertaining to arithmetic in each of the three divisions of the elementary school.

In recent years the problem of preparation of teachers has been simplified by the creation of the "consolidated" or "single" curriculum. According to this plan a teacher is "prepared" to teach at any grade level from the kindergarten through the twelfth grade. "A good teacher can teach anywhere." "He teaches children and not subject matter," according to the advocates of this "new" curriculum.

The consolidated curriculum is the outgrowth of poor planning in teacher education. Because of the great shortage of teachers during and after the recent war many students entered teachers' colleges and prepared for teaching in the secondary schools while there was still a shortage of teachers in the elementary schools. Once this demand for high-school teachers was met, the colleges were faced with the problem of providing for teaching jobs which are to be found in the elementary school. Naturally an easy way to meet the problem was to offer a single curriculum. The proponents of the plan suggest a program of "general"

training in elementary education. The teacher in the elementary school completes a few general courses in "methods" or "principles" of teaching and in "human growth and development." The "study of children," or some such title identified with the area of child development, gives the kind of background essential for successful teaching in the elementary school.

The "consolidating" feature of the single curriculum is especially detrimental to the preparation of the elementary-school teacher. He must satisfy the minimum requirements of one or more "teaching fields" in the high school. He then uses whatever time that remains to prepare for the work of the elementary school. This means that the "new" curriculum has no place in its program for education of students in the nature of

TABLE 3

PROGRAMS FOR TRAINING TEACHERS OF ARITHMETIC IN STATE TEACHERS' COLLEGES

Type of Program	Number of Colleges	Percentage
Same for all teachers	84	65
Differentiated	45	35
Kindergarten-primary grades (Kindergarten and Grades I–III)	45	35
Intermediate grades (IV–VI)	20	16
Advanced grades (VII–VIII)	17	13
General elementary (Kindergarten and Grades I to VIII or IX)	30	23

arithmetic and in the ways pupils must learn it. We may undertake to prepare teachers for the elementary school in the "consolidated" curriculum or we may undertake to prepare them to teach arithmetic and other social arts in a balanced program. We cannot do both at the same time.

Specialization at a particular level does not mean that the teacher should limit his study to the problems which are likely to occur in the grades for that division of the curriculum. He should have an over-all picture of the work at different stages of the program but should be best qualified to deal with those problems which appear at the level of specialization.

Although it may be desirable to have the curriculum differentiated for the training of teachers, this practice has not become the modal pattern of operation. Table 3 shows that 65 per cent of the state teachers' colleges offer only the general curriculum. The differentiation is not carried beyond the third grade in more than half of the cases in a specialized curriculum. The prevailing procedure for differentiation is to offer a course for the primary grades and one for the general elementary grades.

Background in Mathematics for Entering Teachers' Colleges

The number of subject-matter courses a teacher of arithmetic needs depends upon the amount of work done in mathematics before entering college. If a student takes no mathematics beyond eighth-grade arithmetic, he needs more subject-matter courses in mathematics in his college program than a student who has had three or four years of secondary mathematics. The minimum requirement in credits earned in mathematics at the secondary level is one factor in deciding upon the kind of basic courses which constitute minimum essentials in a training program for a teacher of arithmetic.

TABLE 4

NUMBER OF YEARS OF HIGH-SCHOOL MATHEMATICS REQUIRED FOR ADMISSION TO STATE TEACHERS' COLLEGE CURRICULUMS PREPARING TEACHERS OF ARITHMETIC

Kind of Mathematics	Number of Colleges	Per cent
No mathematics	93	76.0
Some form of mathematics	31	24.0
One year of algebra	2	1.6
One year of geometry	2	1.6
One year of both algebra and geometry	5	3.9
One year of general mathematics	2	1.6
One year of any kind of mathematics	18	13.9
Two years of any kind of mathematics	2	1.6

Table 4 shows the number of years of mathematics required of students who enter state teachers' colleges. Over three-fourths of the colleges demand no mathematics of any kind for admission to a curriculum which prepares the student to become a teacher of arithmetic. It is reasonable to assume that many students are admitted who have had no mathematics beyond arithmetic at the eighth-grade level. In that case the student has not studied mathematics for at least four years; hence, much of the subject is forgotten.

According to Carpenter, "No mathematics is required for graduation from high school in California, which accounts for the fact that approximately 50 per cent of the high-school students in Los Angeles graduated without having had any mathematics in the senior high school."[2] Not all of the teachers' colleges in California have admission standards which require the student to present credit in mathematics. Since half of the graduates of high school in this state take no mathematics at the second-

[2] Dale Carpenter, "Planning a Secondary-Mathematics Curriculum To Meet the Needs of All Students," *Mathematics Teacher*, XLII (January, 1949), 45.

ary level, it is almost certain that many of these students enter teachers' colleges without an adequate background for professional training in the teaching of arithmetic. The situation described in California is not unique. The fact that most teachers' colleges require no mathematics for entrance is indicative of the prevailing practices in the country.

Need for a Background Course in Mathematics in Teachers' Colleges

It was shown that most state teachers' colleges admit students who have had no mathematical preparation beyond arithmetic at the eighth-grade level. Students with such meager training are almost certain to have a limited knowledge of arithmetic. Taylor (**15**:10) reported the results obtained from the use of the C. N. Mills Selective Test on the Fundamentals of Arithmetic given to entering Freshmen in Illinois state teachers' colleges for a number of years. The test consists of twenty questions of which six are two-step problems, three are one-step problems, and eleven are examples in the fundamental operations with integers and common and decimal fractions. Allowing a credit of 5 for each correct answer, the average score made by 2,097 Freshmen in five teachers' colleges in one year was 60. In 1934 the average of 209 Freshmen was 65.9, and in 1935 the average of 145 Freshmen was 57.1.

The lack of arithmetic achievement of high-school grdauates was shown on tests in the fundamental processes during the training period of the recruits in the recent war. The low scores made by the naval personnel was the basis for the famous "Nimitz letter"[3] in which he gave a scathing rebuke to the teachers of mathematics for their inability to secure better results in the subject. The writer's experience with preparing textual material in mathematics for naval aviation cadets, all of whom graduated in the upper half of their classes at high school, showed that these boys, although recently graduated from high school, lacked an effective working knowledge and understanding of arithmetic. Since different investigations lead to the conclusion that graduation from high school is not an indication of mastery of arithmetic, there is need of a course in the teacher's training program which will supply a background in this subject. The emphasis of the course should not be on the computational phase but on mathematical principles which govern operations in the basic processes.

Background Mathematics in Teachers' Colleges

It is not possible to state the best kind of course for a background in mathematics for teachers of arithmetic. This course should not be re-

[3] *North Central Association Quarterly*, XVI (January, 1942), 222–23.

stricted to arithmetic. There are many topics in other forms of mathematics which should be understood by the intelligent citizen as well as the teacher of arithmetic. The "Final Report of the Commission on Postwar Plans of the National Council of Teachers of Mathematics" (5:4) drew up a list of twenty-nine topics in mathematics which may be regarded as essential elements of the education of every citizen. The committee expressed the view that a knowledge of these topics is necessary in order to deal with problems of everyday affairs. The teacher of arithmetic should have a minimum knowledge of mathematics which would enable him to check each item in the list. It is certain that he would need to take more mathematics than is offered in the first eight grades in arithmetic in

TABLE 5

NUMBER OF SEMESTER HOURS OF BACKGROUND MATHEMATICS IN THE CURRICULUM FOR TEACHERS OF ARITHMETIC IN STATE TEACHERS' COLLEGES AND THE PERCENTAGE OF COLLEGES HAVING THIS REQUIREMENT

Curriculum	Number of Colleges	Number of Semester Hours		Percentage Requiring Background Mathematics
		Maximum	Average	
Kindergarten-primary grades. . .	45	8	1.4	33
Intermediate grades. .	20	6	1.2	40
Advanced grades.... . .	17	6	1.5	47
General elementary grades..... .	114	8	2.0	65

order to become familiar with all of the twenty-nine topics in this basic list.

A course in background mathematics should constitute part of the training for a teacher of arithmetic. Table 5 shows the number of teachers' colleges which require a course of this kind. The table shows that two-thirds of the colleges offering a curriculum for kindergarten and primary grades do not require a course in background mathematics. In more than half of the colleges offering curriculums which prepare teachers for the intermediate and advanced grades, a background course in mathematics is missing. The situation in the general elementary curriculum is more favorable than for the other specialized curriculums.

The average number of semester hours of background mathematics is very low in each of the four curriculums given in Table 5 because of the large number of colleges which demand no mathematics of this kind. From the list of 114 colleges which have a general elementary curriculum, there are 40 colleges which require no background mathematics. If the

colleges requiring no background mathematics are not considered, the average number of semester hours for the four curriculums in the order given in Table 5 is 3.3, 3.3, 3.3, and 2.7, respectively. These averages indicate that many colleges require more than three semester hours of background mathematics. There was only one college which required more than six semester hours in this field, but many colleges required a year's course in basic mathematics for a total of six semester hours.

Kind of Subject Matter for a Background Course in Mathematics

A background course in mathematics for elementary-school teachers must help the student understand number and its uses if the course is to fill a need in his training program. Conventional college Freshman math-

TABLE 6

KIND OF SUBJECT MATTER GIVEN IN BACKGROUND COURSES IN MATHEMATICS IN 129 STATE TEACHERS' COLLEGES

Description of Course	Number of Colleges	Percentage
Predominantly seventh- and eighth-grade arithmetic	13	10
Advanced arithmetic at high-school or college level	17	13
General mathematics	26	20
Conventional college mathematics	5	4
Combination of method and subject matter	16	13
No description given	4	3
No background course required	48	37

ematics, such as college algebra and trigonometry, does not meet the need of the prospective elementary teacher. In chapter xii, Newsom discusses the mathematical needs of the teacher of arithmetic. A background course should fit into the kind of program he discusses.

The questionnaire used in this investigation showed the nature of the courses taught in this field in state teachers' colleges. Table 6 indicates the kind of courses offered. It is refreshing to note the small percentage of colleges which require conventional college mathematics in the program for training of elementary teachers. A course in general mathematics or in advanced arithmetic at the high-school or college level offers the kind of subject matter that should prove to be functional for the teacher of arithmetic.

The nature of the background course can be determined to a large extent from the textbook used. Table 7 lists the name of each text that was used in at least two colleges. There were twelve other texts that were used but once.

The textbook most widely used is the most recent one published in this

field at the time of this investigation. This text by Buckingham gives the student an opportunity to study the mathematical principles of arithmetic and to use them in the solution of examples and problems. A student who masters the subject matter of a text of this kind has an excellent background for pursuing a professional course in the teaching of the subject. The writer looks with disfavor upon the practice of using the seventh- or eighth-grade book of an arithmetic series for a basic text in a background course. This procedure was followed in a few cases and justified on the ground that the text is used in the schools in which the student will teach. The underlying philosophy back of a program of this kind

TABLE 7

BASIC TEXTBOOKS USED IN BACKGROUND COURSES IN MATHEMATICS IN STATE TEACHERS' COLLEGES

Textbook	Number of Colleges
Buckingham: *Elementary Arithmetic: Its Meaning and Practice*	12
Taylor: *Arithmetic for Teacher-training Classes*	10
Boyer: *Mathematics for Teachers*	8
Newsom: *Introduction to College Mathematics*	4
Richtmeyer and Forest: *Business Mathematics*	3
Betz: *Basic Mathematics*	2
Brueckner and Grossnickle: *Mathematics We Use*, Book II	2
Cooley, Gans, Kline, and Wahlert: *Introduction to Mathematics*	2
Kokomoor: *Mathematics in Human Affairs*	2
Stone, Mills, Mallory: *Unit Mastery Arithmetic*, Grades VII–VIII	2
Twelve other texts which had a frequency of one each	12

necessarily assumes that arithmetic is a tool subject, and the ability to solve the problems in a textboook is the chief goal of such training.

Robinson made a study of examinations given for teachers' licenses in New York City. Also, he observed many teachers of arithmetic in the classroom in the first eight grades in the elementary school. He concluded:

> Elementary-school teachers have at best only a mechanical knowledge of arithmetic even though they are fairly proficient in their skills in the manipulation of its various mechanical processes. Anything like a clear and versatile knowledge of the fundamental principles of arithmetic and their mathematical significance is all but totally lacking on the part of such teachers. Because of this fact, it is difficult to see how the arithmetic they teach or would be called upon to teach to elementary-school children could be anything more than mechanical (**12**:180).

A committee working under the auspices of the National Commission on Teacher Education and Professional Standards sensed the value of background mathematics for all teachers. At the Bowling Green Conference in 1948, the committee report dealing with basic mathematical knowledge, understandings, and appreciations concluded as follows:

All teachers, irrespective of grade levels at which they teach, should have basic courses in mathematics which will give them the opportunity to acquire backgrounds which will enable them to take their places in the world, to broaden their capacities for living happy, useful lives, and to develop those social concepts essential to a full understanding of our democratic way of life. Every teacher should have an appreciation of the basic nature of mathematics and a knowledge of its origin, its general development, and its application and interpretation in everyday life (**18**:150).

The writer agrees with the committee's recommendation that, "a general course in mathematics on the college level should be required of all prospective teachers."

Training in the Teaching of Arithmetic

One major function of a teachers' college is to steadily raise the level of competency of beginning teachers of subjects offered in the curriculum of the elementary school. Since arithmetic always has been part of a basic core in the curriculum, it follows that the student preparing to teach in the elementary school should be required to take a course in the teaching of arithmetic. This plan was in operation about two decades ago. According to Buckingham, "No institution attempts to train elementary-school teachers without one or more special courses in arithmetic" (**3**:319). Unfortunately, this same situation no longer prevails. Table 8 shows that, according to data assembled for the purposes of this study slightly over one-third of the colleges do not require a course of this kind in the general elementary curriculum. Moreover, the individual reports show that very few teachers' colleges require more than three semester hours in a course dealing with the teaching of arithmetic, although the maximum reported is six semester hours.

Table 8 also shows that instruction in the teaching of arithmetic for general elementary grades may be part of a general methods course in almost one-third of the cases. Then, too, in almost half of the colleges which require a course of this kind, the department of education provides the instruction in the subject. The fact that the course frequently is part of a general methods course and that it may be given by teachers in the department of education instead of by teachers of mathematics indicates the lack of importance attached to the subject of arithmetic in a student's program of training to become an elementary teacher.

It is unusual to find the instructor of a course in general methods who is a specialist in professional subject matter in several fields. This teacher may know one field well and devote a major portion of the course to that subject. Many teachers' colleges included in this investigation offer a course in general methods in the skill subjects, including reading, arithmetic, spelling, and handwriting. The teacher may know the professional subject matter in reading but not in arithmetic. In that event the only instruction his students receive in arithmetic will be of a general nature, such as pertains to its social uses and applications. On the other hand, if

TABLE 8

AVERAGE NUMBER OF SEMESTER HOURS REQUIRED AND THE DISTRIBUTION OF SPECIFIC AND GENERAL COURSES IN TEACHING OF ARITHMETIC AT DIFFERENT GRADE LEVELS IN THE DEPARTMENTS OF EDUCATION AND MATHEMATICS IN 129 STATE TEACHERS' COLLEGES

Curriculum	Average No. of Semester Hours Required	Number of Institutions Offering Instruction					
		Separate Course in Teaching Arithmetic		Instruction Given as Part of General Methods	Course Taught by		Total for "Required and Not Required"
		Required	Not Required		Education Dept.	Mathematics Dept.	
Kindergarten-primary grades......	1.4	27	18	8	12	15	45
Intermediate grades.	1.9	14	6	7	6	8	20
Advanced grades....	1.9	12	5	5	4	8	17
General elementary grades...........	1.5	75	39	34	40	35	114

he knows the field of arithmetic, his students may not acquire a proper understanding of the other subjects. In either case, a program of this kind is unsatisfactory for preparing teachers for the elementary school.

Robinson gave a dark picture of the professional qualifications of teachers of arithmetic in teachers' colleges when reporting an investigation made about twenty years ago. Since that time the professional standards of staff members in teachers' colleges have been raised considerably. On the other hand, some of the criticisms he directed against the poorly trained teachers of arithmetic in professional schools are still applicable to the teachers of courses in education who give instruction in arithmetic. Robinson's conclusion is summarized in the following paragraph.

Teachers have been employed to give professional courses in arithmetic, who in their undergraduate and graduate preparation did not confine such prepara-

tion to the field of mathematics nor to any considerable extent to the subject of arithmetic. Therefore it might be said that the majority of teachers of courses in arithmetic in the professional schools for teachers have come to their positions either by chance or for reasons other than those of specific preparation in the field. Such a practice is difficult to justify wherever it exists in educational administration and especially difficult to justify when followed in the administration of a professional school for teachers (**12**:180).

The practice of telescoping the teaching of arithmetic into a course of general methods and of having the course taught by an instructor who may be poorly trained in mathematics makes it impossible to train competent teachers of arithmetic. There is a core of technical subject matter in arithmetic which makes it necessary to offer a course in this subject.

TABLE 9

YEAR IN WHICH PROFESSIONAL COURSE IN ARITHMETIC IS GIVEN IN STATE TEACHERS' COLLEGES

Year of College	Number of Colleges	Year of College	Number of Colleges
First. .	6	First or second. .	2
Second. .	13	Second or third.	2
Third. . .	29	Second or fourth.	5
Fourth.	10	Third or fourth.	11
		Not indicated. . . .	9

This course should enable the teacher to acquire skills and techniques which are peculiar to the teaching of arithmetic. It is only as a teacher acquires specific techniques and skills of this kind that he becomes a professional worker.

Table 9 shows that the third year is the modal year in which the professional course is given. In a four-year program this seems to be the proper place for it. The fact that this course is given in some colleges in the first or second year is probably due to a limited training period of only one or two years in those colleges.

A course in the teaching of arithmetic should be given after the background course has been completed and before the student does his practice teaching. The background course usually is offered in the first or second year. In the latter situation it is necessary to have the professional course in the Junior or Senior year.

Textbooks Used in Professional Courses in the Teaching of Arithmetic

The basic textbook used in a course in the teaching of arithmetic gives some indication of the nature of the course. A total of thirteen different

texts were reported in use in the colleges included in this study. Table 10 gives a list of the texts which have a frequency of two or more. The three most frequently used texts are by Brueckner and Grossnickle, Morton, and Spitzer. The text by Buckingham is not adapted to the requirements of a basic professional book on the teaching of arithmetic. Buckingham states in the preface that his book is a subject-matter text and does not deal with methods. This text should be used in a background course in arithmetic but not in a professional course in the teaching of the subject.

TABLE 10

TEXTBOOKS USED IN PROFESSIONAL COURSES DEALING WITH THE TEACHING OF ARITHMETIC

Textbook	Number of Colleges
Brueckner and Grossnickle: *How To Make Arithmetic Meaningful*	27
Morton: *Teaching Arithmetic in the Elementary School,* Volume I, II, or III.	21
Spitzer: *The Teaching of Arithmetic*..	8
State Syllabus on the Teaching of Arithmetic.	4
Taylor: *Arithmetic for Teacher-Training Classes*.	4
Buckingham: *Elementary Arithmetic: Its Meaning and Practice*	3
Roantree and Taylor: *Arithmetic for Teachers*	2

Amount of Mathematics Required in the Training of Teachers

Table 5 gives the number of semester hours of background mathematics required for graduation from the teachers' colleges participating in this study. Table 8 gives the number of semester hours in the teaching of arithmetic. Table 11, which is a combination of Tables 5 and 8, gives the total number of semester hours of mathematics required for graduation from these colleges.

Table 11 shows that one-third of the colleges offering a curriculum for kindergarten and primary grades do not require any courses in mathematics or in the teaching of arithmetic, and that 10 per cent of the colleges offering a curriculum for the general elementary grades do not make mathematics a requirement for graduation. It is difficult to understand how a professional school which has for its main objective the preparation of elementary-school teachers would fail to emphasize specific training in the teaching of one of the core subjects. In this respect the professional preparation of teachers of arithmetic has retrogressed within the past two decades. Evidence was cited to show that in 1930, at the time of the publication of the National Society's last yearbook dealing with arithmetic, all professional schools offered at least one course in arithmetic.

Table 11 shows that the average number of semester hours of mathematics required in the four curriculums in teachers' colleges used in this study varies from 2.8 to 3.5. This means that the average number of semester hours in both background and professional courses is equivalent to about one course in arithmetic for one semester. A meager offering of this kind is inadequate to equip the teacher to do a satisfactory job in the teaching of arithmetic, especially since most of the colleges have no mathematical requirement for entrance from high school. As long as this situation exists in the training of teachers of arithmetic, the schools of our country are certain to be staffed with teachers who are not adequately trained to teach arithmetic except as a tool subject. When this philosophy is accepted, drill is the chief instrument in the teacher's professional kit.

TABLE 11

AVERAGE TOTAL NUMBER OF SEMESTER HOURS OF MATHEMATICS REQUIRED BY STATE TEACHERS COLLEGES FOR COMPLETION OF CURRICULUM FOR TEACHERS OF ARITHMETIC

Curriculum	Average Number of Semester Hours	No Mathematics Required		Number of Colleges
		Number	Percentage	
Kindergarten-primary grades.	2.8	15	33	45
Intermediate grades. .	3.1	3	15	20
Advanced grades...	3.4	0	0	17
General elementary grades.....	3.5	11	10	114

Comparison of the Amount of Arithmetic Required in Teacher-training Programs Today and in Those of Twenty Years Ago

Although the modal length of the program for training of elementary-school teachers during the past twenty years has increased from two years to four years, the amount of time given to instruction in arithmetic has not increased. New subjects have been added to the curriculum in the intervening period. Besides, more of the so-called cultural subjects as given in a liberal-arts college have been added to the curriculum in teachers' colleges. As a result of these changes, no more time is now assigned to professional instruction in arithmetic in four-year programs than was formerly required in two-year programs.

Taylor made a study of the amount of arithmetic required in the training programs of teachers in 1928 compared with the requirements in the same subject in 1935. Table 12 gives the results of his study. He concluded that there was a marked decrease in the number of semester

hours required in the training program of elementary-school teachers over the period of about seven years.

Table 12 indicates that striking changes occurred in all four curriculums in the percentage of schools that did not require courses in arithmetic. In three of these curriculums there was a marked increase in the percentage of schools not requiring any courses in arithmetic, but in the general elementary curriculum the opposite trend was noted. A comparison of the median number of semester hours of arithmetic for the years 1928 and 1935 shows that there has been a decrease in three of the four curriculums. Table 11 gives the average number of semester hours where Table 12 gives the median number. For approximate comparison, the two measures may be considered equivalent. The median number of semester hours for 1928 is about the same as the average number of semester hours today, indicating that there might have been a slight increase in the amount of the required work in mathematics since 1935. Taylor's article mentions only arithmetic. If he did not consider general mathematics as offered in background courses, there has been a continued decrease in the amount of arithmetic required in the training program of elementary teachers since 1928. The substitution of general mathematics

TABLE 12

THE PERCENTAGE OF TEACHER-TRAINING INSTITUTIONS REQUIRING SPECIFIED NUMBERS OF SEMESTER HOURS IN ARITHMETIC IN 1928 AND 1935 IN FOUR KINDS OF CURRICULUMS*

Number of Semester Hours Required	Kind of Curriculum							
	Kindergarten-Primary		Intermediate Grades		Advanced Grades		General Elementary	
	Percentage of Schools		Percentage of Schools		Percentage of Schools		Percentage of Schools	
	1928	1935	1928	1935	1928	1935	1928	1935
0	20.6	29.7	10.2	28.3	9.9	51.4	45.8	14.8
1–1.9	6.3	6.3	0.8	5.7	2.5	2.9	0.0	11.1
2–2.9	35.7	33.3	26.3	31.1	28.1	17.5	25.2	25.9
3–3.9	11.1	14.5	22.9	11.3	7.4	8.7	11.9	7.4
4–4.9	15.9	9.0	22.0	16.0	33.0	11.6	3.4	11.1
5–5.9	4.8	5.4	9.3	4.7	10.7	4.9	6.8	7.4
6–Up	5.6	2.7	8.6	2.8	8.3	2.9	6.8	22.2
Median	2.8	2.4	3.5	2.5	4.1	1.0	2.2	3.0

* Adapted from Taylor (15:13).

for arithmetic per se seems, to the writer, highly desirable. The most unfavorable part of the program has been the increasing percentage in the number of teachers' colleges which do not require mathematics in the preservice training of elementary teachers. If the central tendency in the number of semester hours remained about the same since 1928 while the percentage of schools not requiring mathematics increased, some colleges must have increased the required number of hours of mathematics. Later in this chapter an analysis will be made to determine the relationship between the geographic location of colleges and the number of semester hours of mathematics required for certification of a teacher of arithmetic.

TABLE 13

STANDARDS OF ACHIEVEMENT REQUIRED FOR GRADUATION IN THE TRAINING PROGRAM OF TEACHERS OF ARITHMETIC IN 129 STATE TEACHERS' COLLEGES

Activity	Number of Colleges
For those who took no background mathematics:	
Must attain norms for Grade VIII on standardized tests in arithmetic	14
Must make a satisfactory score on a test prepared by the mathematics department	12
No check given on a student's competency in arithmetic.	22
For those who took background mathematics:	
Must attain norms for Grade VIII on standardized tests in arithmetic	10
Must make a satisfactory score on a test prepared by the mathematics department	21
Satisfactory completion of background course is sufficient.	50

Competency in Arithmetic Demanded of Students in Teachers' Colleges

At several places in this chapter it was pointed out that some teachers' colleges do not require mathematics for graduation from curriculums designed for elementary-school teachers. A student who did not take mathematics in high school might become an arithmetic teacher without having courses in the subject after completing the eighth grade. The questionnaire used in this study asked whether or not the college checked on a student's competency in arithmetic before he was graduated. This applied to all colleges, not merely to those which did not require background courses in mathematics. Table 13 presents a summary of these findings.

A study of the table shows that a greater number of colleges which require a background course in mathematics also demand a satisfactory achievement score on a standardized test in arithmetic or a test prepared by the mathematics department than is the case with colleges which do

not prescribe background requirements in mathematics. The ability of a prospective teacher to make a satisfactory score on a test of this kind is not, in itself, dependable evidence of his ability to teach arithmetic. On the other hand, if his achievement in arithmetic is below the norm of a standardized test at the eighth-grade level, he is inadequately trained for teaching the subject. Table 13 shows that a large number of teachers' colleges, besides requiring a course in background mathematics, attempt to maintain a minimum standard of competency in the subject. Unfortunately, too many colleges neither offer a background course in mathematics nor require a minimum standard of competency in arithmetic. Inspection of the individual reports received revealed the fact that, of all the colleges which demand no mathematics of any kind for graduation, only two require the student to show that he has a working knowledge of arithmetic as a tool subject equivalent to that of the average eighth-grade pupil. This unfavorable situation must be corrected before acceptable minimum professional standards can be established for teachers of arithmetic.

Two National Reports on the Training of Teachers of Arithmetic

Two national agencies published reports on the training of arithmetic teachers at about the same time in the autumn of 1947. The first is known as the *President's Manpower Report* and the other is the final report of the *Commission on Postwar Plans of the National Council of Teachers of Mathematics.* Although the two agencies had worked independently, each of the reports stresses the need for specific training in the teaching of arithmetic for teachers in the elementary school.

The report of the President's Scientific Research Board emphasized the need for better preparation of teachers of arithmetic in these words:

> A professionalized subject-matter course emphasizing the use of mathematics in projects undertaken by children to learn the meaning of concepts is a minimum requirement for the training of mathematics teachers for the first six grades. Such a course should demonstrate the proper use of laboratory materials and of appraisal techniques and should give the prospective teacher a clear notion of specific objectives for each grade level.
>
> The training of the mathematics teacher for Grades VII and VIII is an entirely different matter. A vast amount of new material has come into the work of the pupils in these grades during the last thirty years. This material is not ordinarily included in the traditional sequential courses of the high school or of the college. Consequently, too many beginning teachers of seventh- and eighth-grade mathematics are expected to teach without adequate time for planning that which they themselves have never studied in systematic fashion. The teacher of the seventh and eighth grades should have a minor consisting of mathematics courses especially designed to meet his needs. Few colleges now

offer enough mathematics of this type for elementary teachers. The content of such courses is a matter that should be determined by a comprehensive study. In the meantime every teacher-training institution should try to make available a course in the teaching of mathematics for these grades (8:99).

The recommendations presented in the report issued by the National Council of Teachers of Mathematics listed more specific things to accomplish or to provide for than appeared in the Manpower Report. Some of the suggestions of the National Council's commission are summarized in the following paragraph:

> If you are planning to teach in the elementary school, three things are very important: (1) a good methods course; (2) at least one course which will give you subject-matter background for the mathematics taught in Grades I through VI; (3) at least one course that will give you subject-matter background for teaching general mathematics of Grades VII and VIII. Junior high school teachers of mathematics should have at least a minor in college mathematics. This should include a year of general mathematics in college, a course in statistics, a course in the mathematics of investment, and a professionalized course in the subject matter taught in these grades (5:11).

Each of these reports stressed the need for a good course in the teaching of arithmetic. The commission of the National Council also called for a course in background subject matter for the first six grades. Both reports gave special consideration to the kind of specific preparation beyond the training for the first six grades needed by the teacher in Grades VII and VIII. Current practice in the training of elementary teachers in teachers' colleges does not follow the recommendations of these reports. Out of 129 teachers' colleges there were only 17 which differentiated the curriculum for these grades. The more common procedure is to provide training to teach in the first eight grades in the elementary school.

Regional Accrediting Associations

Regional and national accrediting associations recommend certain standards for certification of teachers. There are six regional accrediting associations in this country. However, in the listing below, there are only five groups, as the Western Association, which includes only the state of California, is combined with the Northwest Association. The following list gives the names of the states in each grouping. The state of Montana is divided between the Northwest Association and the North Central Association. In this study, Montana is included in the Northwest Association.

New England Association:
: Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, and Vermont

Middle States Association:
Delaware, Maryland, New Jersey, New York, and Pennsylvania
Southern Association:
Alabama, Florida, Georgia, Kentucky, Louisiana, Mississippi, North Carolina, South Carolina, Tennessee, Texas, and Virginia
North Central Association:
Arizona, Arkansas, Colorado, Illinois, Indiana, Iowa, Kansas, Michigan, Minnesota, Missouri, Nebraska, New Mexico, North Dakota, Ohio, Oklahoma, South Dakota, West Virginia, Wisconsin, and Wyoming
Northwest and Western Associations:
California, Idaho, Montana, Nevada, Oregon, Utah, and Washington

TABLE 14

MINIMUM REQUIREMENT IN YEARS OF TRAINING FOR CERTIFICATION OF ELEMENTARY TEACHERS IN STATES GROUPED ACCORDING TO REGIONAL ASSOCIATIONS

Association	Number of States	Years of Training				No Requirements
		1	2	3	4	
New England. .	6	0	2	0	3	1
Middle States. . .	5	0	0	0	5	0
Southern...... .	11	1	6	0	4	0
North Central..... .	19	3	8	3	5	0
Northwest and Western...	7	1	2	1	3	0
Total. . .	48	5	18	4	20	1

It may be of interest to determine whether or not the regional location of colleges has an influence on the amount of mathematics required for certification of teachers of arithmetic. Returns from the questionnaire used in this study provide the data for this comparison.

Table 14 gives the minimum number of years of training required for certification of elementary-school teachers in the states identified with the different regional associations. All the states in the Middle States Association demand a minimum of four years of training for elementary-school teachers. This is not true of any of the other associations. The standards for the New England Association rank second. Massachusetts, of the New England Association, does not have any state requirements for certification as each school district sets its own standards. In practice, most of the elementary-school districts in this state require a bachelor's degree for certification.

Thirty of the forty-eight states are included within either the Southern or the North Central accrediting association. It is in these states that the standards for certification are lowest. The minimum requirement for cer-

tification of an elementary-school teacher is two years or less in over 50 per cent of the states in each of these associations. A minimum of one year of training is required in about one-sixth of the states in the North Central Association. To a large extent these minimum requirements for certification are not enforced at the present time because of the shortage of teachers. It is apparent that the professional training of many elementary teachers in these regions is inadequate and definitely substandard. The table shows that the training demanded for certification of elementary-school teachers varies markedly in different sections of the country.

TABLE 15

NUMBER OF STATE TEACHERS' COLLEGES HAVING DIFFERENTIATED CURRICULUMS AND NUMBER OF COLLEGES REQUIRING MATHEMATICS FOR ADMISSION, GROUPED ACCORDING TO ACCREDITING ASSOCIATIONS

Association	Number of Colleges	Differentiated Curriculums Provided		Mathematics Required for Admission	
		Number	Percentage of Group	Number	Percentage of Group
New England	11	5	45	4	36
Middle States	28	10	36	8	29
Southern	20	2	10	3	15
North Central	61	25	41	12	20
Northwest and Western	9	3	33	2	22
Number	129	45	....	29	....
Per cent	....	......	35	....	22

Table 15 gives the distribution of the teachers' colleges included in this study among the different accrediting associations and shows the extent to which mathematics is required for admission to college and differentiated curriculums are provided for teachers in training in the institutions comprising each regional group. The ratio of the number of colleges which have the same curriculum for all students preparing to teach in elementary schools to the number which have differentiated curriculums is almost 2:1, when all of the 129 colleges are considered.

The colleges of the New England Association are almost equally divided in this respect, but in the Southern Association the uniform curriculum prevails over the differentiated types in the ratio of 9:1. Earlier in this chapter it was found that the modal tendency in differentiation of the curriculum is to provide one curriculum for kindergarten-primary grades and another for the general elementary grades. A curriculum which offers specialization in a restricted-grade area, such as for the kindergarten-primary grades, should enrich the training of a teacher in this field. The

practice of providing almost no differentiation, as is current in the Southern Association, cannot provide adequate training in the teaching of arithmetic in all grades for all teachers in the time devoted to the subject. In some of the states in this association, there is not much differentiation between the curriculum for elementary- and secondary-school teachers. When the curriculum of a teachers' college fails to differentiate for a particular type of training it assumes the function of a general college or of a college of liberal arts. The tendency of many teachers' colleges to change their name from a teachers' college to a state college or to some other more general title indicates that such colleges are interested in broadening their offerings for college training, but it may also presage the de-emphasizing of their function as professional schools for the training of teachers.

Table 15 also shows the number of colleges in each regional group which prescribe mathematics as an entrance requirement. The highest percentage of teachers' colleges requiring mathematics for entrance is shown for the institutions in the two regional associations that have the highest standards for certification. The North Central and Southern Associations, which have the lowest standards for certification, likewise have the smallest percentage of colleges which require mathematics for admission. This may be interpreted to mean that, in general, students may enter teachers' colleges in these two regions with less background in mathematics and leave college with fewer years of training for teaching in the elementary school than in the other regions of this country. The amount of training required in mathematics in the different regions will be considered further.

Mathematics Required in Teachers' Colleges Grouped According to Regional Accrediting Associations

The amount of mathematics required for admission to teachers' colleges and the length of the training program in these colleges vary in different geographic regions. In view of these variations, it seems advisable to consider the implications of differences in the amount of mathematics these colleges require of students who will be certified as qualified to teach arithmetic in the elementary school.

Table 16 gives the total number of semester hours of mathematics required in teachers' colleges grouped geographically. This table shows that the teachers' colleges in New England and the Middle Atlantic States Associations require much more mathematical training than the teachers' colleges in the other associations. In the general elementary curriculum, twice as many semester hours of mathematics in the Middle States group and three times as many in the New England group are required as by

those in the Northwest-and-Western group. This great disparity between two sections of the country in the amount of training given in preparation for teaching one of the core subjects in the curriculum of the elementary school is unjustifiable. These variations indicate that no definite standards or goals have been established for the training of teachers in arithmetic.

Preparation of Teachers of Arithmetic in Liberal-Arts Colleges and in Colleges of Education

Elementary-school teachers may receive their training in a state teachers' college, in a college of education of a university, or in a depart-

TABLE 16

AVERAGE NUMBER OF SEMESTER HOURS OF MATHEMATICS INCLUDED IN THE TOTAL PROGRAM OF TRAINING REQUIRED FOR TEACHERS OF ARITHMETIC BY TEACHERS' COLLEGES GROUPED ACCORDING TO ACCREDITING ASSOCIATIONS

Associaton	Number of Colleges	Average Number of Semester Hours in Different Curriculums			
		Kdg.-Primary	Inter-mediate	Advanced	General Elementary
New England....	11	$5.4_{(5)}$*	......	. .	$5.8_{(11)}$
Middle States...	28	$4.8_{(10)}$	$2.0_{(1)}$	. ... .	$3.8_{(28)}$
Southern.........	20	$3.0_{(2)}$	$5.0_{(2)}$	$5.0_{(2)}$	$3.2_{(19)}$
North Central.....	61	$1.7_{(25)}$	$2.9_{(17)}$	$3.1_{(15)}$	$3.2_{(47)}$
Northwest and Western.	9	$1.7_{(3)}$	. ..	. ....	$1.9_{(9)}$
Average for All†...	....	2.8	3.1	3.4	3.5

* The subscript indicates the number of colleges offering a particular curriculum. The table is read as follows: Of the 11 colleges identified with the New England Association, 5 offer the kindergarten-primary curriculum, and the average number of semester hours of mathematics in this curriculum is 5.4.

† These amounts are taken from Table 11; hence they may not agree exactly with the computed averages in the above table due to rounding off the above data to the nearest tenth.

ment of education of a liberal-arts college. We have already discussed the program of training as it is found in state teachers' colleges. Next, we shall examine briefly the requirement in mathematics in the other two kinds of colleges which prepare teachers for the elementary school.

The writer found it inexpedient to make an exhaustive questionnaire study of the training program in colleges of education and in departments of education in liberal-arts colleges. Instead, he selected seven states at random and studied the catalogues of the colleges in these states. It was decided to select at least one college from each of the five regions denoted by the accrediting associations as listed in Tables 15 and 16. The states selected were Massachusetts, Pennsylvania, North Carolina, Ohio, Ne-

braska, Colorado, and Washington. An examination of 142 catalogues of colleges in these states showed that no program for the training of elementary-school teachers is offered in 62 of these colleges. The remaining 80 colleges, consisting of 18 colleges of education and 62 departments of education, offer training for elementary teachers. The amount of mathematics required in these 80 colleges is given in Table 17.

Table 17 shows that only 12 out of 62 departments of education in liberal-arts colleges require a course in the teaching of arithmetic. One of

TABLE 17

AVERAGE NUMBER OF SEMESTER HOURS IN BACKGROUND MATHEMATICS AND IN THE TEACHING OF ARITHMETIC REQUIRED IN DEPARTMENTS OF EDUCATION OF LIBERAL-ARTS COLLEGES AND IN COLLEGES OF EDUCATION FOR PREPARATION OF ELEMENTARY TEACHERS IN SEVEN STATES SELECTED AT RANDOM AS DETERMINED FROM A STUDY OF THEIR CATALOGUES

Kind of College	No. of Colleges	Average No. of Semester Hours of Background Mathematics	Number of Colleges Requiring Background Mathematics	Average No. of Semester Hours of Teaching of Arithmetic	Number of Colleges Requiring Course in Teaching of Arithmetic	Total Semester Hours of Mathematics Required	No. of Colleges Offering General Course in Teaching in Elementary Schools
Liberal arts (Education department).........	62	0.1	1	0.5	12	0.6	31
College of education.... .	18	0.5	3	1.2	9	1.7	6
Total... .	80	0.2	4	0.7	21	0.9	37

these 12 colleges requires a course in background mathematics which is conventional Freshman mathematics of a semester of college algebra followed by a semester of trigonometry. Since only twelve of these colleges require a course in the teaching of arithmetic, the remainder, 50, do not require the student to take a course of this kind. There are 31 colleges in this group which do not offer a course of any kind in arithmetic. In the remaining 19 colleges the student may elect a course in the teaching of arithmetic to have it contribute toward the accumulation of a given number of required credits in elementary education. Thirty-one of these departments of education offer a course in general methods in teaching in the elementary school.

The offerings in the field of arithmetic in colleges of education are superior to those in the departments of education in liberal-arts colleges.

Half of the colleges of education demand a course in the teaching of arithmetic, and one out of six requires a course in background mathematics. Table 17 shows that the average number of semester hours in mathematics required in colleges of education is 1.7 as compared with an average of 0.6 for liberal-arts colleges. In each case the amount of training in the subject is too meager to be effective.

Frequently criticisms have been made of the program in normal schools and teachers' colleges for offering too many methods courses and not enough subject courses. The program for the training of elementary teachers in liberal-arts colleges seems to have corrected that error. Unfortunately, these colleges have pitched out the baby with the bath. Unless a teacher acquires specific skills and techniques to do a particular job, he cannot properly be classified as a professional worker. The so-called cultural courses of a liberal-arts college may enable the student to rate high on a scale of the social graces, but this student cannot become a competent teacher of arithmetic or of any other subject in the elementary school until he understands the underlying philosophy back of the teaching of that subject. It is related that General Patton commented on the training of men for the Army as follows: "I don't give a d—— if a man says 'ain't' instead of 'doesn't.' I want men who can shoot." The teacher should have a general cultural background, but at the same time he must have a knowledge of specific skills which are essential in the work of his profession. If the sample of the education departments of liberal-arts colleges and of the colleges of education is representative of all of these colleges, only about one out of four requires any kind of training which specifically fits the teacher to do a professional job in the teaching of arithmetic. Standards for certification of teachers which will permit such neglect of professional training are in dire need of revision.

Summary

This investigation, made from reviewing the literature on the subject, questionnaire returns from 129 state teachers' colleges, and a study of college catalogues, deals with the program for certification of teachers of arithmetic in the elementary school. The findings are as follows:

1. Twenty states require a bachelor's degree for certification of elementary-school teachers. About twenty years ago, at the time of the National Society's last publication dealing with arithmetic, the modal length of the program of training of elementary-school teachers was two years.

2. The acute shortage of teachers in the elementary school makes it necessary to issue substandard licenses to many teachers. The minimum standards set forth in this investigation are not followed in practice at present.

3. About twice as many states require an average of more than two and a half times as many semester hours in such subjects as history and the social studies and in science as in mathematics for certification of elementary-school teachers.

4. The same curriculum is prescribed for students who are preparing to teach in different grades of the elementary school in about two-thirds of the state teachers' colleges. The curriculum in the other third of the colleges offers a program of training for teachers in kindergarten and primary grades as well as a program for the general elementary grades. About 15 per cent of all teachers' colleges offer a differentiated program for teachers of Grades IV to VI and of Grades VII and VIII.

5. A student does not need any credits in mathematics at the secondary level to be admitted to about three-fourths of the state teachers' colleges.

6. The curriculum in one-third of the state teachers' colleges offering a program for kindergarten and primary grades requires an average of 1.4 semester hours of background mathematics. The curriculum in about two-thirds of the colleges offering a program for the general elementary grades requires an average of 2.0 semester hours of background mathematics. The state teachers' colleges in states located in the territory of the New England and the Middle States accrediting associations require an average of about three times as much background in mathematics as those in the Northwest and Western Associations and much more than those teachers' colleges in the Southern and North Central Associations.

7. A course in general mathematics is the predominant type of subject matter offered in required background mathematics in state teachers' colleges.

8. A student is not required to take a course in the teaching of arithmetic in at least one-third of the state teachers' colleges. In almost half of the colleges requiring a course in the teaching of arithmetic, the course is part of a general methods course, and it is taught by a member of the education department and not by a member of the mathematics department.

9. The average total requirement in mathematics to graduate from a state teachers' college is about three semester hours. About 33 per cent of the students in the curriculum for kindergarten and primary grades and 10 per cent in the curriculum for the general elementary grades may graduate without taking mathematics courses of any kind. A range in the average total number of semester hours of mathematics is from 1.9 for the colleges in states located in the area of the Northwest and Western accrediting associations to 5.8 for the colleges in the region of the New England Association. The average requirements in mathematics for the state

teachers' colleges within the boundaries of the New England and the Middle States Associations are much higher than those for the colleges in the other associations.

10. Although the modal length of the training program for elementary teachers during the last twenty-five years increased from two years to four years, the amount of training required in mathematics decreased during that interval.

11. Only about half of the state teachers' colleges which require no background course in mathematics check on the student's competency in arithmetic. A greater percentage of the colleges which require a course in background mathematics also require the student to pass a satisfactory achievement test in arithmetic than is the case with colleges which have no background requirements in mathematics.

12. A study of the catalogues from a sample list of 142 liberal-arts colleges and universities in seven different states showed that 80 of these institutions had departments of education or colleges of education which prepare teachers for the elementary school. From a list of 62 departments of education, only one requires a course in background mathematics and only twelve require a course in the teaching of arithmetic. The average total requirement in mathematics for these 62 colleges is 0.6 semester hours. The average requirement in mathematics in 18 colleges of education is 1.7 semester hours, or about half the average requirement in state teachers' colleges.

Recommendations

The data assembled in this investigation show that the program for training of teachers of arithmetic in some states is on a much higher plane than in other states. Since these conditions exist, the recommendations for the training of teachers must be of two kinds: First, those which should become effective within a year or two; second, those which should become effective within the next five or ten years.

Immediate changes in the program for the training of teachers of arithmetic are as follows:

1. A teacher should not be certified to teach arithmetic in the elementary school who has not had at least one good course in the teaching of the subject. This applies equally to those students who take their training in a liberal-arts college and to those who prepare in a one- or two-year curriculum of a teachers' college.

2. Those students who are admitted to a teachers' college or a liberal-arts college and have had no mathematics beyond eighth-grade arithmetic should be required to take a course in background mathematics before they are permitted to take a course in the teaching of arithmetic.

3. The teacher of both the professional course and the background course in arithmetic should have specific training in arithmetic.

The training of teachers in the elementary school will always be inadequate until the minimum requirement for certification is a bachelor's degree or its equivalent. Since the present length of the modal period of training of elementary teachers is four years, it will be assumed that a program of this kind offers the minimum acceptable training for a teacher of arithmetic. The other recommended minimum requirements for teachers of arithmetic are as follows:

1. The course of training should be differentiated so as to prepare teachers for the kindergarten and the first six grades in one group and teachers for Grades VII and VIII in another group. Teachers preparing for rural areas should be trained to teach arithmetic in the first eight grades, with the chief emphasis on the work of the first six grades.

2. Every elementary-school teacher, including teachers of kindergarten and special subjects, should have a course in background mathematics, preferably six semester hours in length. This course should acquaint the student with the basic principles of the number system, introduce him to both formal and informal geometry, to algebra as a system of generalized number, and to the study of trigonometry as used in indirect measurement.

3. All teachers in the elementary school, except those who are preparing to teach fine and applied arts, music, or in other specialized fields, should have a course in the teaching of arithmetic in which both social applications and mathematical meanings of the subject are treated for the different grade levels covered in the program of each teacher. This course should not be part of a general methods course.

4. Teachers of arithmetic for Grades VII and VIII should have at least a minor in college mathematics. Besides, they should have a course in teaching the subject at this level.

5. The teacher of the professional course in the teaching of arithmetic should be specifically trained in arithmetic. There is a greater possibility that this condition will prevail in teachers' colleges if the mathematics department rather than the department of education is responsible for the course.

6. Students in liberal-arts colleges who wish to become teachers of arithmetic in the elementary school should be required to meet the same standards of preparation as those who are trained in teachers' colleges.

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CHAPTER XII

MATHEMATICAL BACKGROUND NEEDED BY TEACHERS OF ARITHMETIC

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Introduction

General Comments

A curriculum designed for the training of elementary-school teachers, in common with the rest of our educational program, is a subject for debate. The writer, by virtue of his position that is concerned in part with teacher education in the state of New York, is aware of the force with which proponents of rival philosophies can suggest changes in the curriculum. It is very difficult, and perhaps impossible, to construct a teacher-training program that will receive any unanimity of acceptance. However, such chapters as the present one must be written in order that specific proposals may be set up for purposes of discussion and trial.

As Grossnickle has stated in chapter xi, a large number of colleges offering curriculums for the training of elementary-school teachers include no specific background course in mathematics. Of those institutions offering courses in background mathematics, it is common to require only a three-hour or a six-hour course. Courses on the teaching of arithmetic are offered in most institutions, and in other ways experience in arithmetic is provided the prospective teacher; thus more actual arithmetic is being taught in many curriculums than a superficial examination of college bulletins would indicate. Nevertheless, it is the writer's considered opinion that it is possible and desirable to include in the curriculum for the training of elementary-school teachers a minimum of a three-hour course of background mathematics, in addition to a course on the teaching of arithmetic; moreover, a six-hour course is preferred. All too frequently teachers in the elementary grades are hardly a jump ahead of their alert students, and many teachers have confided in the writer that they lack confidence before their classes in approaching various arithmetical concepts. A student will excuse a teacher who must look up a question pertaining to geography, but he becomes critical of the instruc-

tor who stumbles in dealing with arithmetic. Moreover, it must be emphasized that an incorrect presentation by the elementary-school teacher of concepts in arithmetic may handicap a student for the rest of his life; many secondary-school and college teachers have been forced to labor at great lengths to rectify arithmetical misunderstandings that students have been taught on more elementary levels.

The content of a course in background mathematics is a matter of genuine disagreement on the part of even those who are acknowledged authorities in the field. Thus, the ideas expressed in this chapter must be regarded as essentially those of the writer, although the treatment as given is the result of considerable thought, of consultation with many people, and of participation in many conferences concerned with the subject. In general, the trend in such a course as we are considering will be set by the textbooks that are available. Fortunately, some very fine books are now in print; the titles of some of them appear in the bibliography at the end of the chapter.

It is the writer's belief that an adequate treatment of the concepts advocated in this chapter requires a minimum of two years of high-school mathematics. As Grossnickle states, such a requirement for admission to the elementary-training program is not universal. However, it seems imperative that teacher-training institutions should require at least two years of high-school mathematics of all prospective elementary teachers. If students not having the prescribed two years of secondary mathematics are admitted to an elementary teacher-training program, there seems to be no alternative other than the requirement of an additional course providing the equivalent of two years of high-school mathematics, offered as a prerequisite to the background course that is the subject of this chapter.

The course proposed in this chapter for prospective teachers of elementary arithmetic is a course in arithmetic. Although it might be the hope of some educators, especially mathematicians, that the course should comprise additional material of considerable significance to modern mathematics and science, and to our culture in general, such a position seems unrealistic at the present time. The course should be restricted to the kind of arithmetic that will actually be taught by the teacher plus the background development that will make the arithmetic meaningful and will cause the teacher to be certain of his position as an expositor of arithmetical concepts. The final report of the Commission on Postwar Plans of the National Council of Teachers of Mathematics[1] contained a

[1] *Guidance Pamphlet in Mathematics for High School Students.* Final Report of the Commission on Postwar Plans of the National Council of Teachers of Mathematics. New York: *Mathematics Teacher,* 1947.

check list of twenty-nine questions in mathematics, a positive answer to which is regarded as virtually a requirement for successful dealing with the problems of everyday affairs. It would indeed be fine if such a list of questions could be regarded as forming an essential part of the outline of the course under consideration, but it is doubtful that such is possible, certainly not in a three-hour course and probably not in a six-hour course.

Some Important Factors To Be Considered in the Development of the Course

In the development of the proposed course, it seems essential to keep in mind certain fundamental factors. First, the student entering the course possesses already a collection of isolated ideas of an arithmetical nature; however, it is probable that he is unaware of the fact that mathematics is a human invention, that it is the most perfectly organized knowledge in existence, that it is the result of one of man's most serious struggles to control his environment and to make a society of men possible. Second, pure mathematics is an abstract science. However, students taking the course in question do not possess the maturity to comprehend the abstract structure of arithmetic, so there should be a conscious effort at all times to relate all concepts to tangible things. Moreover, at the earliest possible moment the processes of arithmetic should be treated in terms of their actual utility. The beginning college student in general is not content with hearing statements about the usefulness of concepts; he wants to be a participant; he wants to apply his knowledge to actual problems. Third, a truly significant background course in mathematics, as with any subject, should demand breadth of study and exploration on the part of the student. Many courses in college arithmetic can be condemned for too rigid adherence to a textbook. Arithmetic is a live subject for investigation on the part of modern scholarship. Mature students of mathematical pedagogy as well as research mathematicians are turning out voluminous materials, a considerable part of which can be understood by and will be of interest to the prospective elementary teacher. This chapter will note certain references, perhaps unfamiliar to some instructors in college arithmetic, that have been influential in the viewpoints of the writer. Fourth, even though the course that is proposed is not one in methods, the approach of the instructor will be reflected strongly in the ideas put into practice when the student becomes a teacher. Thus, it behooves the instructor to employ methods at all times that are marked by clarity of exposition and accuracy of statement. It is the writer's impression, as the result of considerable observation within college classrooms, that an amazingly large number of teachers of a back-

ground course in mathematics are quite unprepared to provide the leadership and inspiration that is essential.

The Historical Development of Arithmetic

Some Purposes of the Historical Treatment

It is doubtful that there would be any general disagreement with the premise that the prospective teacher of arithmetic must have some understanding of the evolution of arithmetical concepts and notations. The historian knows that the development in antiquity of various cultures basic to our own is intimately associated with the evolution of mathematical ideas. In fact, an effective historical introduction to the subject of arithmetic will reveal to the novice that the development of civilization has depended heavily upon the availability of an adequate mathematics. The student should gain a better understanding of the present utility of arithmetical concepts by observing the evolution of such concepts as the result of man's struggle to make possible certain accomplishments.

References

The bibliography contains several references that will provide the students and the instructor of a course in college arithmetic excellent background material for studying the historical foundation of arithmetic. Extensive and up-to-date references will be found in the sixth edition of the well-known outline by Archibald.[2] It seems appropriate also to call attention to the recent readable and informative book by Ore.[3] The first chapter of this book is especially important for the early part of the course under consideration; the history of number symbols associated with various cultures is given in some detail in this chapter, including plates of finger symbols and tallies, and of Egyptian, Greek, Chinese, and other numerals. Of course, the historical treatment will draw heavily on the well-known writings of Smith, Sanford, Cajori, Karpinski, and others. Every student of college arithmetic should have the experience of observing the symbolism and the arithmetical methods employed, for example, in the *Rhind Papyrus*.[4] The exposition of this Egyptian mathematical document of nearly 4,000 years ago contains photographic plates of the original papyrus, its free translation, and a commentary. An excellent

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[2] R. C. Archibald, *Outline of the History of Mathematics*. No. 2 of the Slaught Memorial Papers. Buffalo, New York: Mathematical Association of America, University of Buffalo, 1949.

[3] Oystein Ore, *Number Theory and Its History*. New York: McGraw-Hill Book Co., 1948.

[4] *Rhind Mathematical Papyrus*. Buffalo, New York: Mathematical Association of America, University of Buffalo, 1927 (2 vols.).

treatment of the growth of number and the operations pertaining to it will be found in the comparatively new book by Boyer.[5] Numerous other specific references might be given, according to the emphasis given the course by the instructor; for instance, the number systems of the American Indian provide an interesting insight into the use of mixed bases.[6]

Number

Arithmetic began with the efforts of primitive man to make a record of and to convey information about his possessions. The very fundamental notion of a one-to-one correspondence, that led primitive man to the procedure of counting, is equally basic today in the higher reaches of mathematics. Moreover, as Victor F. Lenzen has said, "The fundamental quantitative procedure of science is counting a set of objects in order to characterize it by a number."[7] In other words, basic considerations in regard to the actual nature of number itself deserve thorough consideration.

The necessity for some written record of the numbers obtained by counting led to the development of systems of notation. There were the early, unwieldy systems based on a variety of principles, until some talented Hindu had the idea in about 500 B.C. that led to our present Hindu-Arabic system of notation. The struggle of man before he conceived of a zero and the fact that the symbols for the digits have become stable in only recent centuries make an illuminating study for students.[8] The writer well remembers the time in his teaching career when he discovered that a majority of students in a Freshman class believed that the Hindu-Arabic system had emerged at some point in history in finished form; the suggestion was made by one student that the system appeared on a table of stone dropped from the heavens.

The Real Number System

Positional Notation

The introduction of the Hindu-Arabic number system, frequently known now as the common system, leads immediately to a consideration of positional notation in the construction of a numeral system. That is,

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[5] L. E. Boyer, *An Introduction to Mathematics for Teachers.* New York: Henry Holt & Co., 1945.

[6] H. C. Whitener, "The Number System of Three Southwestern Indian Tribes," *The Pentagon,* Vol. 2. Albion, Michigan: Albion College, Fall, 1942.

[7] V. F. Lenzen, *Procedures of Empirical Science,* Vol. I, No. 5. *International Encyclopedia of Unified Science.* Chicago: University of Chicago Press, 1938.

[8] D. E. Smith and Jekuthial Ginsburg, *Numbers and Numerals.* New York: Bureau of Publications, Columbia University, 1937.

the student needs to understand very clearly what is meant when we say that the value of every digit in a positional notation depends upon the position which it occupies. An adequate consideration of this point forces one to a consideration of number bases and, in the opinion of the writer, to a consideration of number bases other than ten. The student is not in a position to understand many of his own difficulties in dealing with numbers until he has had experience with bases other than ten; likewise, he will be in a much better position to comprehend the troubles of his own students when he becomes the teacher if he has fought essentially the same battles in dealing with a strange base.

The point of view just expressed was confirmed in a striking way by an experiment reported by Darrell Holmes.[9] Holmes tells of a project carried out in 1948 when students in a junior high school were encouraged to speculate on many things pertaining to possible number systems. This was followed by the construction of an experiment which matched an experimental class against a control class to attempt a tentative answer to the question, "Do the children who have studied different number systems better understand our own system?" The conclusion of the study, briefly stated, was that "within the limitations of the sample and the variables controlled, the children of the experimental group are superior to those of the control group in their knowledge of numbers and number systems—including our own."

The student should be encouraged to change a numeral written to the base ten to a numeral written, for example, to the base eight or thirteen. Of course, in the former case, there will only be needed the digits 0, 1, 2, 3, 4, 5, 6, 7; in the latter case all the traditional symbols will be needed plus three new ones that the student must manufacture. The game becomes especially interesting when the student uses none of the common symbols but manufactures a completely new set. Such experiences not only will emphasize some fundamental elements pertaining to numerical systems but will serve to make realistic the statement that "mathematics is a human invention." The student will be surprised and impressed when he learns that some number bases other than ten have been advocated strongly for certain purposes. The base twelve is in actual use by certain estimators of capacity and has arithmetical advantages.[10] The base two is indispensable in certain parts of advanced, theoretical mathematics; moreover, certain modern computing machines can operate only when all numbers employed are transformed into the binary system. Several

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[9] Darrell Holmes, "An Experiment in Learning Number Systems," *Educational Research Bulletin* (Ohio State University), XXVIII (April 13, 1949).

[10] F. E. Andrews, *New Numbers*, pp. 42–44. New York: Harcourt, Brace & Co., 1935.

writers have emphasized the importance of the base two in the analysis of certain games.[11,12]

In examining such a numeral as 239, written to the base ten, the student should be expected to realize that the notation is a convenient symbol for $2(10)^2 + 3(10) + 9$. It is not expected that a teacher of arithmetic in the elementary grades will discuss exponential forms, but the student of college arithmetic has less than an adequate treatment of numerals if he does not realize the significance of the statement that every numeral written to the base ten is merely a simple notation for an algebraic polynomial in powers of ten. In fact, the student should be ready to write 239 as $2(10)^2 + 3(10)^1 + 9(10)^0$. To facilitate the discussion, the course should include a short review of exponents.

The Properties of Integers

The properties of the integers (or natural numbers), the operations with integers, and the fundamental laws of arithmetic pertaining to the operations on integers are considered with care by several of the more popular textbooks upon college arithmetic. However, such studies could be supplemented in an effective manner by studying, for example, the first part of Klein's famous work.[13] Klein, one of the most distinguished mathematicians of the first part of the present century, received international acclaim as a teacher of mathematics. A very fine, brief treatment has also been given by Courant and Robbins in the first part of their excellent book, that should be in every reference library.[14]

A natural number itself is an abstraction. The number five, for example, is an abstraction associated with the fingers on one hand, the toes on one foot, and the collection of pennies that may be exchanged for a nickel. That is, the number five is associated with any collection of objects having the property that there is a one-to-one correspondence between the individual objects of the collection and, for example, the fingers on one hand; the number five is described as the count of such a collection. As indicated earlier in this chapter, the notion of one-to-one correspondence is most important and deserves repeated emphasis. By study-

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[11] *American Mathematical Monthly,* XXV (1918), 139–42.

[12] H. D. Larsen, "Dyadic Arithmetic," *The Pentagon,* Vol. 1. Albion, Michigan: Albion College, Fall, 1941.

[13] Felix Klein, *Elementary Mathematics from an Advanced Standpoint.* (Translation from the third German edition by E. R. Hedrick and C. A. Noble.) New York: Macmillan Co., 1932.

[14] Richard Courant and Herbert Robbins, *What Is Mathematics?* pp. 1–9. New York: Oxford University Press, 1941.

ing correspondences and by proper definition of concepts, it is established that a collection of six objects is greater than a collection of five objects; so six is regarded as a greater number than five. By a continuation of such procedures it is established that the natural numbers form an ordered collection. The most significant part of this discussion, however, is that even the mature mathematician does not hesitate in his deliberations to move from considerations involving abstract numbers to the related considerations involving collections of objects.

The addition of two and three, very simply, is taken to be the count obtained when two mutually exclusive collections made up of two objects and three objects, respectively, are put together. This may be shown in visible fashion by working with pieces of chalk, or perhaps by the use of dots, as do Courant and Robbins. The latter authors continue to a study of multiplication, using a tangible illustration involving collections that is easy to follow and has a sound, logical basis. With little delay, the inverse operations, subtraction and division, can be defined. All students of arithmetic should know by name the commutative laws of addition and multiplication, the associative laws of addition and multiplication, and the distributive law. These laws become quite meaningful when one considers the boxes of dots, employed by Courant and Robbins. Of course, a similar scheme may be employed by using pieces of chalk or other objects. As indicated by Courant and Robbins, the properties of zero may be established intuitively by considering the role it must play in the studies just outlined.

The Four Basic Arithmetical Operations

After the fundamental nature of the four basic arithmetical operations has been established, it is the writer's experience that considerable study, including drill, must be devoted to each one of the operations. The concern now is with the actual mechanics involved in carrying out the operations. In the paragraphs that follow, detailed suggestions are given for the treatment of addition; somewhat similar outlines can be developed for the other operations.

The study of addition can be introduced by reference to ancient counting boards and the abacus. In fact, it is expected that there will be frequent comparisons with early arithmetical techniques whenever present-day procedures are examined. The modern device employed when adding numbers may be regarded as having its basis in the algebraic addition of polynomials. One might start with such an algebraic addition as:

$$\begin{array}{r} 5x^3 + 7x^2 + 4x + 9 \\ 6x^2 + 8x + 5 \\ \hline 5x^3 + 13x^2 + 12x + 14 \end{array}$$

Then for the addition, 5749 + 685, the desired result follows readily as shown below:

$$\begin{array}{r} 5(10)^3 + 7(10)^2 + 4(10) + 9 \\ 6(10)^2 + 8(10) + 5 \\ \hline 5(10)^3 + 13(10)^2 + 12(10) + 14 \end{array}$$

Of course, $14 = 1(10) + 4$, so $1(10)$ should be added to $12(10)$, thereby providing $13(10)$. But, $13(10) = (10 + 3)(10) = 1(10)^2 + 3(10)$. The term $1(10)^2$ may be added to $13(10)^2$, thereby providing $14(10)^2$. But, $14(10)^2 = (10 + 4)(10)^2 = 1(10)^3 + 4(10)^2$. As a concluding step, then, $1(10)^3$ may be added to the original first term on the left, giving $6(10)^3$. In summary, the desired sum is $6(10)^3 + 4(10)^2 + 3(10) + 4$, or, when written in our common notation, the result is merely 6434. Obviously, the purpose of the reduction just pursued is the elimination of all coefficients of ten or more in the original polynomial obtained for the sum. In practice, the whole process of reduction is carried out rather effectively by "carrying"; in fact the student should observe that the usual method of "carrying" accomplishes the reduction that is desired. The problem just worked in detail might be repeated in the case of many additions, some of which should involve more than two addends.

Many teachers believe it is desirable to consider, in addition to the usual method of addition, such processes as the Hindu method, the scratch method, and perhaps several so-called short methods. Probably such considerations serve to emphasize the actual nature of the additive process. Moreover, many college students are too hesitant in their use of the primary addition combinations, so it is desirable to introduce many opportunities for giving students needed practice with them.

No treatment of addition would be complete without giving consideration to various types of checks. Students should be encouraged early in their careers to regard the checking of an answer as an essential part of the solution of any problem. Obviously, the first step in checking any result involves the use of intuitive, or common-sense, considerations. In other words, students should be taught to be skeptical of answers that appear to be unreasonable. A systematic checking of a sum may be accomplished by such a device as adding in the opposite order from that employed originally.

A very important method of checking addition, as well as the other operations, involves "casting out nines." To be specific, a necessary condition for any sum to be correct is that the excess of nines in the sum must be equal to the excess of nines in the sum of the excesses in the addends. The procedure of casting out nines is of considerable interest to students, and it is intimately related to the choice of ten as the base of our number system; hence, the method deserves serious examination. A

very important theorem that is basic in the scheme of casting out nines is that the excess of nines in a given number is equal to the excess of nines in the sum of its digits.[15] Without going into a formal proof of the theorem, it can be made plausible by such a consideration as the following:

$$\begin{aligned} 462 &= 4(100) + 6(10) + 2 \\ &= 4(99 + 1) + 6(9 + 1) + 2 \\ &= [4(99) + 6(9)] + (4 + 6 + 2) \\ &= (\text{a multiple of } 9) + (\text{sum of the digits}). \end{aligned}$$

Very early in the study of each one of the operations, the student of college arithmetic should have an opportunity to apply his knowledge to problems arising from actual situations. This should be done even though the problems may be of an extremely elementary nature. Some students, without additional training, are so inaccurate in their use of arithmetic that they have genuine difficulty checking entries and withdrawals against the balance shown on a bank statement.

It should be apparent to the reader how similar outlines can be developed for the treatment of subtraction, multiplication, and division.

The Fractions

Common Fractions. The consideration of fractions follows readily after the study of division. It is apparent that there is no integer satisfying the definition of 1 divided by 2; that is, no integer times 2 gives 1. Thus, it becomes necessary to expand the number system at this point if it is desired that all problems in division have an answer. Students must realize that the situation just encountered is a typical difficulty in dealing with the concept of number, that the evolution of number ideas has been related intimately to the demands of the various operations. In the case of 1 divided by 2, the new number created is one of the fractions; it has the name, one-half, and is written conveniently as $\frac{1}{2}$. Of course, in a presentation to elementary students a fraction may be described as one or more of the equal parts of a unit, or, perhaps, as one of the equal parts of several units. Basically, however, one must not forget the fact that a fraction is a new number, created to augment the integers.

Fractions are bothersome to a large number of people, and it is the opinion of the writer that extraordinary thoroughness and patience must be employed in their presentation. A large amount of special terminology has been developed in connection with fractions, some of it unnecessary and of false connotation, but it seems desirable to encourage students to become familiar with a considerable part of it. For instance, there are such terms as common fractions, simple fractions, complex fractions,

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[15] H. S. Hall and S. R. Knight, *Higher Algebra*, pp. 62–63. London: Macmillan Co., 1936.

proper and improper fractions, mixed numbers, lowest terms, decimal fractions, binary fractions, and so on. The student should realize early in his experience with fractions the fundamental nature of the principle that the value of a fraction is not changed if both numerator and denominator are multiplied by the same number or if they are divided by the same number. An important fact, frequently slighted in courses in arithmetic, is that every integer may be written as a fraction; in other words, there is a more general classification, known as "the class of rational numbers," that comprises both the integers and the new numbers, the fractions. The rational numbers are sometimes described as the numbers of elementary arithmetic, and they form a very important component of the so-called real number system.

When the fractions are introduced into arithmetic it must be realized that many techniques and principles promulgated for integers are not necessarily appropriate for the new numbers. That is, the mathematician must virtually start over and reintroduce the operations of addition, subtraction, multiplication, and division. Any new principles introduced at this point must provide results consistent with those obtained in previously accepted considerations involving the integers, but this fact is about the only guide that is available. Such studies provide the instructor with an excellent opportunity to explain the nature of mathematical knowledge, especially the role of consistency in mathematical endeavor. The task of introducing and treating the fractions in systematic fashion is far from simple, and reference to scholarly thought on the subject is desirable.[16] The writer has no rigid point of view in regard to the operation which should be introduced first in the study of fractions but would prefer the following order: multiplication, division, addition, subtraction. All of these operations deserve considerable attention, and there should be an adequate amount of drill on basic principles.

Decimal Fractions. A decimal fraction, by definition, is a fraction that has a denominator which is a power of ten. The number $\frac{431}{1000}$, therefore, is a decimal fraction. As a matter of interest, however, this fraction may be written in the form $\frac{400}{1000} + \frac{30}{1000} + \frac{1}{1000} = \frac{4}{10} + \frac{3}{100} + \frac{1}{1000} = 4(10)^{-1} + 3(10)^{-2} + 1(10)^{-3}$. In a similar manner any decimal fraction may be written as an expansion in negative powers of ten wherein the coefficients are less than ten. It seems obvious and desirable, therefore, to expand the concept of positional notation already considered to include the writing of decimal fractions. Thus, we write:

$$732\frac{431}{1000} = 7(10)^2 + 3(10)^1 + 2(10)^0 + 4(10)^{-1} + 3(10)^{-2} + 1(10)^{-3}, \text{ or } 732.431.$$

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[16] Klein, *op. cit.*, pp. 28–31.

The point between the 2 and 4, called a decimal point in the decimal system of notation, is employed in this country as the separatrix, that is, as the mark which separates the integral portion of a given number from the part that has a value less than one. The student should realize that there is no world-wide agreement upon the symbol for the separatrix; thus, the number just given would be written in England as 732·431 and in France and Germany as 732,431. The comma as used in this country to segregate digits in blocks of three is a typically American custom. No presentation of the common method of writing decimal fractions would be complete without giving the student suggestions for reading the numbers; many persons are uncertain as to acceptable procedures in this regard.

The methods of performing the operations on decimal fractions should receive justification by referring back to the techniques employed on common fractions. A corollary of this statement is the fact that a fundamental problem in connection with the study of decimal fractions pertains to the writing of a decimal fraction as a common fraction to lowest terms and the rewriting of any common fraction in the form of a decimal fraction. Adequate drill must be provided at this point. There should be thorough consideration of the rules employed in the location of decimal points in the results given by the various operations; the study is incomplete unless the reasons behind the rules are clear. Students will face the problem of rounding off numbers when they are asked to obtain certain answers to the nearest tenth or perhaps thousandth; this situation arises especially in division where so many quotients of two integers involve an infinite number of digits in the decimal representation. Thus, a preliminary consideration of rounding off a number should be provided at this point.

The studies of the previous paragraph lead naturally to the consideration of the very important statement that any rational number, when expressed in decimal form, can be written either with a finite number of digits or in the form of a repeating decimal. The converse of this statement is also true. An adequate treatment of this topic involves the consideration of infinite geometric progressions. The writer has found that students with two years of high-school mathematics can handle infinite geometric progressions with a minimum of difficulty and with a maximum of comprehension and enjoyment.[17] The study of the proposition pertaining to repeating decimals virtually forces one to some mention of infinite decimals like π, where there is no regular repetition of groups of digits.

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[17] C. V. Newsom, *An Introduction to College Mathematics*, pp. 126–27. New York: Prentice-Hall, Inc., 1946.

It seems entirely appropriate to devote part of a class hour to irrational numbers and their importance. Of course, the rational numbers and the irrational numbers compose the complete system of real numbers.

Probably it is needless to emphasize again the importance of introducing numerous practical problems in connection with the topic of decimal fractions. Our monetary system, the metric system, and many other scales of measurement are built so definitely upon the decimal system that the student must have a sound working knowledge of the subject.

The study of decimal fractions leads immediately to a consideration of percentage. In fact, it should be emphasized that percentage is not a new subject but is merely a continuation of the study of fractions. The student should develop ease in changing a per cent to its equivalent common fraction or decimal fraction, or vice versa. The consideration of applied problems should be fundamental in the presentation.

The Arithmetic of Measurement

Introduction

The writer has observed with considerable chagrin that courses intended for the preparation of teachers of elementary arithmetic continue to ignore the fundamental importance in the knowledge of everyone of the arithmetic of measurement. In practice, a number originates in two ways, namely, as the count of the members comprising a collection or as a number associated with a measurement. A measurement, by definition, is the process of assigning a number to some property of an object. It is quite true that the process of measurement involves counting, but there are considerations involved in the adoption of a number associated with a measurement with which most people, including mathematicians, seem to be in utter ignorance. Two days before writing this section of the chapter the writer listened with some amazement as his young son in the grades explained a process being taught to him that involved computations with measurements. He started with measurements that were precise to a tenth of a foot and ended up with answers that were precise to a hundredth of an inch. It is a common experience to see teachers before a class use two-digit and three-digit data to obtain a result with six digits. When one stops to realize that most of the numbers dealt with in life are numbers of measurement, and when we consider that the nature of such numbers and the special techniques necessary in performing operations upon them are virtually ignored, it is possible to judge to a certain extent the seriousness of the offense being committed against prospective teachers.

The Process of Measurement

The study of the arithmetic of measurement should be introduced by an examination of the process of measurement itself. A brief but sound analysis of the subject has been provided by Victor F. Lenzen.[18] There should be some understanding of the meaning and significance of personal errors and of instrumental errors. The student must comprehend such terms as precision, accuracy, significant digits, and absolute and relative errors; the first two of these terms are frequently used incorrectly. The terms present difficulties for many students, and it is essential that students analyze each one of a considerable collection of measurements in order that the story being told by each number is thoroughly understood.

Systems of Measurement

It is in connection with this part of the course that it seems appropriate to introduce the various systems that are in common usage for the measurement of such magnitudes as time, length, area, volume, capacity, weight, angles, etc.[19] Special attention should be given in all the systems to the arbitrary nature of the fundamental units employed; such a consideration should serve to indicate further the strong element of human inventiveness to be found in our mathematics and science. Among all the systems of units studied, major attention should be given to the English system of weights and measures and to the metric system.[20] As a part of the study of these systems, students should be given an opportunity to actually perform measurements, keeping in mind principles of precision and accuracy in the writing down of the measurements obtained. The exercises given the students should involve considerable experience in changing from one type of unit to another.

Computations with Approximate Numbers

Numbers of measurement, or denominate numbers as they are sometimes called, are approximate numbers. Many authors have treated with considerable success the subject of operations upon approximate num-

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[18] Lenzen, *op. cit.*

[19] For the true history of the sexagesimal system of circular measure, see O. E. Neugebauer and Others, *Studies in the History of Science*, p. 16. Philadelphia: University of Pennsylvania Press, 1941.

[20] *The Metric System of Weights and Measures*. Twentieth Yearbook of the National Council of the Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1948.

bers; only a few of many possible references can be mentioned.[21,22] In view of the fact that an approximate number merely indicates the range within which the true value will be found, it is only possible in performing a computation upon approximate numbers to obtain the range within which the desired result will be found. Since the analysis may lead to considerable arithmetic, it is customary to employ less-involved procedures that provide results which are fairly reliable. Each author has his own way of outlining such procedures; the following statements have been found to be useful by the writer:

In the addition or the subtraction of approximate numbers, it may be assumed that the result has an absolute error as large as that in any one of the numbers involved. Thus, all numbers should be rounded off to within one place of the position occupied by the last significant digit in the number possessing the greatest absolute error. The final result, then, should be rounded off to the same precision as the number having the greatest absolute error.

In the multiplication or the division of approximate numbers, it may be assumed that the result is no more accurate than the number involved which has the least number of significant digits. Thus all numbers should be rounded off to within one significant digit of that least number. The final result, then, should be rounded off so as to contain the same number of significant digits as the weakest number just mentioned.

Many practical formulas require powers or roots of approximate numbers. So, if time permits, the considerations given above should be extended to powers and roots.

The treatment in this chapter up to this point has been somewhat detailed because of the writer's belief that the material suggested forms a virtual minimum in the training program of prospective teachers of arithmetic. By careful budgeting of time, such a syllabus can be covered in a three-hour course. If a six-hour course is taught, it is conceivable that about two-thirds of the time will be devoted to a somewhat more elaborate study of the material so far presented than is possible in a shorter course. The topics treated henceforth may be regarded as second-semester material.

Further Attention to Applications

Although it is to be presumed that many arithmetical applications have been considered in connection with the material already discussed

[21] A. Bakst, *Approximate Computation,* Twelfth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1937.

[22] C. N. Shuster, "Teaching Computation with Approximate Data," *Mathematics Teacher,* XLII (March, 1949), 123–32. This article also contains an excellent bibliography on the subject.

in this chapter, the work would be incomplete without a systematic study of certain topics that are a natural part of elementary arithmetic. The topics suggested for consideration are indicated below somewhat in their order of importance as background material for the prospective teacher.

Evaluation of Formulas

Adequate attention should be given to the evaluation of formulas, keeping in mind such principles as those that pertain to approximate numbers, checks, and so on. The mensuration formulas should receive major consideration; in fact, students should be expected to know the more important formulas pertaining to area and volume. Informal justifications for the formulas should be developed by the students for class criticism. Excellent suggestions for problems involving the mensuration formulas can be found in such a book as the one by Kern and Bland.[23] Problems employing the Pythagorean theorem should be solved. This will lead to a consideration of square roots; some authorities would allow adequate time for the examination of devices for taking square root, but the writer is more inclined to use that time in the study of tables of square roots and cube roots. In actual practice, roots are extracted by the use of tables, logarithms, computing machines, or the slide rule. Of course, the policy followed in teaching square root to prospective teachers should be related closely to the program followed in that regard in the elementary grades; the writer believes that the traditional method for the extraction of square root should not be taught on any level. Time could be devoted also to simple problems involving the lever, Ohm's law, and other simple principles from science and technology.

Ratio and Proportion

There should be a review of ratio and proportion, with a mention of the terminology and utility of variation. Probably the best-known uses of ratio and proportion pertain to similar triangles; interesting and informative problems can be designed for the discussion of this subject. Most books on college algebra now provide a brief treatment of variation.

Business Arithmetic

It is essential that every prospective teacher of arithmetic understand the concepts basic to the material treated in a traditional course in business arithmetic. The meaning of simple interest and compound interest should be emphasized. The solution of long problems in compound interest should be facilitated by the use of compound interest tables; in fact,

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[23] W. F. Kern and J. R. Bland, *Geometry with Military and Naval Applications.* New York: John Wiley & Sons, 1943.

experience in the use of the tables is highly desirable. Such terms as discount and present value should be introduced and employed in problems. The study of taxation should include the computation of federal and state income taxes under simple, hypothetical conditions. Property, liability, health, and life insurance are topics of urgent importance, and typical policies should be discussed. It is the writer's belief, based on experience, that simple forms of annuities can be considered successfully by students in the course; this suggestion assumes the use of tables. Unfortunately, only minimum attention can possibly be given to annuities, so an excellent elective course for prospective teachers of arithmetic would be, if properly designed, a course in the mathematics of finance.

Statistical Concepts

In view of the growing importance of statistical concepts, some of the common ideas in the field of statistics should be analyzed. Attention should be given to frequency distributions and their graphical representation, including a brief consideration of the normal frequency curve. The only averages treated would be the arithmetical mean, the median, and perhaps the mode. The only measure of dispersion receiving attention would be the standard deviation. It is to be assumed that the instructor of the course would eliminate this statistical material if it is treated adequately in another course.

Probability

Some topics already discussed—certain statistical notions and insurance, for instance—are based on the laws of probability. Even more important, however, probability has become an important part of arithmetic in its own right.[24] It is noteworthy that the study of probability, with the related considerations of permutations and combinations, provides a genuine insight into arithmetical thinking. Thus, a brief introduction to the subject, including both mathematical probability and empirical probability, seems to be desirable. A sufficient background in the field can be given for the purposes of the course by working from fundamental principles; the writer's ideas on this subject have been recorded elsewhere.[25]

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[24] H. L. Levinson, *The Science of Chance.* New York: Rinehart & Co., 1950.

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CHAPTER XIII

IN-SERVICE DEVELOPMENT OF TEACHERS OF ARITHMETIC

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The Concept of Supervision

We have arrived in this country at a new concept of the nature and role of supervision in public education, but so far we are only at the threshold. In school systems throughout the country there is a growing dissatisfaction with the old conception that the role of supervision is that of prescription of content and method and inspection of procedures and outcomes. This traditional view of supervision caused dissatisfaction not only because it is based on an undemocratic and authoritarian system of school administration but also because it has not demonstrated its ability to establish effective methods of improving instruction. No better example of the futility of the "prescribe and inspect" theory of supervision can be found than in the supervision of the arithmetic programs of elementary schools. In spite of the research and publication of findings in the psychology of arithmetic and related fields, and in spite of a large number of improvements in materials for teaching, the program of arithmetic in many elementary schools is not significantly different from the program of schools in the 1890's. There are superficial differences, to be sure, but in fundamental respects the same procedures and the same psychology of learning often obtain.[1]

The new concept of supervision, which is the point of view from which this chapter is written, regards supervision as a process of mutual study and criticism by all the members of the staff of a school or a school system. All schools are faced with various problems which need solution.

[1] Cf. Leo J. Brueckner and Others, *The Changing Elementary School*, pp. 333 ff. The Regent's Inquiry into the Character and Cost of Public Education in the State of New York. New York: Inor Publishing Co., 1939.

Some of the most important of these problems are concerned with instructional method and related matters. The new conception of supervision suggests that the most important supervisory service which can be rendered is that of organizing teachers for the study of problems which exist in their own schools. Teachers are encouraged to identify important problems, study them critically, propose ways of solving them, and appraise procedures and outcomes. When supervision of this kind is carried on, it is possible to involve large numbers of persons in an enterprise in which each person has a stake and with which he can identify himself closely. When these conditions are fulfilled, the teacher becomes a member of a group striving to solve an important problem. His attitude toward the process is quite different from that in which he merely receives a directive from some supervisory or administrative officer.

Obviously, the role of a supervisor working under the new concept is quite different from the traditional role. A person who works with the new conception of supervision regards his most important work as that of organizing and leading programs designed to promote the effective study of important problems. His work is one of furnishing leadership and co-ordination of effort. The need for leadership of this kind is everywhere apparent in American education, and it is idle to suppose that significant work in curriculum improvement will be done by teachers unless they are given such leadership.

Virtually all the major problems related to curriculum and teaching method lend themselves to undertakings of this type. However, it is important in the success of a supervisory program that great care be taken in identifying and setting up problems for study by the staff of a school. Teachers are busy people. Their primary interests are necessarily practical, since it is on them that the responsibility falls for the day-by-day work of the classrooms. On the whole, teachers are not interested in abstract and academic problems. They tend to be concerned not about "the relation of theories of learning to the teaching of arithmetic," but about such problems as, "How can I improve my teaching in the fourth grade so that my children will get a better understanding of and will be more skilful in multiplication and division?" There is no doubt that the improvement desired in the fourth grade is closely related to the psychology of learning, but to the teacher himself it is the fourth grade which is the primary concern, not the psychology of learning.

It is assumed, therefore, that one of the crucial factors in carrying on supervision of this kind is that of identifying problems for study and experiment which teachers regard as important because they see clearly their relationship to the actual life in the classroom and to their work as

teachers. When this condition is met, it is possible to organize fruitful study of instructional problems.

Fortunately, teachers are interested in teaching technique. Many teachers often carry on modest experimentation with techniques and procedures on their own initiative and without any particular encouragement of administrative help. Some teachers try to find new and better ways even at the risk of incurring official displeasure. There is reason to believe, therefore, that if supervision is conceived as a process of mutual study and investigation by the staff of a school or a school system under competent leadership, far more will be gained in the improvement of instruction than if a series of directives on content and method are merely passed out by a supervisory authority.

Studying the Content of Arithmetic

In nothing which is taught in the elementary school is it more important that teachers have an adequate understanding of the content itself than in arithmetic. Yet it is probable that elementary-school teachers have less insight into the content of arithmetic than into any other subject. It is a peculiar fact that people usually do not study arithmetic except in the elementary school, and it must be admitted that mastery there is ordinarily far from thorough. In fact, arithmetic is taught so badly in so many schools that people rarely study any branch of mathematics further than the minimum demanded by requirements for graduation from high school or for a prospective vocation.

It is no wonder, then, that teachers themselves do not know much about the content of arithmetic and that they suppose that ability to cipher constitutes sufficient mastery of the field to enable them to teach children.

One thing which is probably needed is for teacher-training institutions to pay more attention to providing prospective elementary-school teachers with a better understanding of arithmetic and the number system. At the present time very few of them appear to do this, so the responsibility must fall on those responsible for the in-service training of teachers. The question of methods of teaching arithmetic is an important one in both preservice and in-service training, *but the understanding of the subject itself is of at least equal importance.* Teachers will not see the importance of changing their methods of teaching unless they have sufficient understanding of the number system to enable them to see the deficiencies in their methods.

The authors of this chapter suggest two areas of study which will be of value for teachers in service in improving their understanding of the system of number and in consequent improvement of teaching method.

These two areas are (*a*) a study of the historical and social development of arithmetic as a systematic discipline, and (*b*) the application of theories of learning to the teaching of arithmetic. Each of these fields will be discussed briefly.

Familiarity with the history of arithmetic is an asset to a teacher for various reasons. In the first place, those who are competent in any field always have some insight into the origin and development of that field of study. Such an understanding is a necessary part of the general knowledge of an individual in that field. Second, knowledge of the history of arithmetic helps to immunize a person against the current belief that arithmetic is relatively devoid of any significant conceptual content and is merely a "skill" subject made up of mechanical processes. Third, such knowledge has immediate practical value because it is extremely valuable for children to study the history of number on the level of their development, and a teacher, in order to develop such a unit of work, needs a rich background of knowledge himself.

This survey of the historical development of number and its social uses can lay a background for the study of the formal aspects of the number system. The primary aim of such a study should be to give teachers themselves meaningful experience with the number system and insight into what is ordinarily called the "meaning theory" of teaching arithmetic. Teachers should study, or at least review, such concepts as the decimal nature of our number system and its implications for the four fundamental arithmetical operations. It should also involve understanding of the relation of the decimal system to such social uses as measurement, systems of money, and so forth. Such a study of the content of arithmetic should be helpful in demonstrating to teachers that arithmetic is an important branch of the science of mathematics and is not merely a mechanical skill relatively devoid of any significant conceptual matter.

While it is felt that such a study by teachers will be of great benefit in improving the status of arithmetic in a school system, everything depends on how it is carried out. An administrator can hand down an edict that all teachers below a certain grade are expected to read a certain number of pages in a certain text on the history of arithmetic and then be prepared to discuss them at a meeting to be held in two weeks. Such an edict will successfully stimulate derision and subtle sabotage; it is doubted that it will stimulate interest in the historical development of arithmetic.

It must be remembered that supervision of the kind desired includes the furnishing of leadership for the study of important problems. In many situations it would be well at the beginning to keep the number participating in such a study small, restricting it perhaps to the staff of one building or to a part of that staff. If the procedure "catches on" (and

indeed, if it is well done, it will "catch on," for the history of number is fascinating), it can be extended to include others in the school and school system.

Specific techniques for organizing in-service training for teachers in various kinds of school situations are discussed later in this chapter. However, it should be noted that interest is as important in the education of adults as in the education of children. The skilful supervisor is one who knows how to arrange educative experiences for people in terms of their interests, abilities, and previous experience.

Throughout the program of studying the history of arithmetic there should be frequent application of concepts and materials to the immediate work of the teacher. Unless this is done, and unless those participating see concrete instances of value, they will tend to lose interest and to regard the whole process as just another academic undertaking without real significance for the work of the teacher.

Applying Theories of Learning to the Teaching of Arithmetic

The other aspect of study which is suggested for a supervisory program in arithmetic is that of applying what is known about the learning process to teaching method and content. Such a study, if carried on in the right way and in accord with the right concept of supervision, can be of immediate service in improving the teaching of arithmetic.

It is well to recognize first that the majority of teachers are not interested in the psychology of learning, per se. It may be a matter of regret to some, but it is the truth, nevertheless. Teachers *are* interested in such problems as: "Why can't my children be more accurate in their computation?" "Why can't sixth graders solve the written problems?" "What is the best book to use for third-grade arithmetic?" "Should we use arithmetic workbooks in our school?" "Is the grade placement of topics in our course of study in arithmetic good?" These are not primarily questions about the psychology of learning, but psychology has something to contribute to the solution of each. The important thing is that they are problems which are important to teachers.

In planning for learning experiences for children, we have long recognized the principle that we begin with a person where he is and build from there. This principle holds quite as firmly for adults. It is proposed that a supervisory program begin with the problems of teachers in teaching arithmetic and that relevant findings in the psychology of learning be introduced to assist in the solution of these problems. One of the roles of the supervisor should be to help teachers identify the important problems which they face and to assist them in bringing to bear on these problems all of the resources available.

When an in-service program, organized along these general lines, is developed in a school system, supervision becomes the united efforts of administration and staff to improve conditions for learning.

Assisting Teachers To Initiate Learning Activities Designed To Improve Instruction in Arithmetic

In developing an in-service program in arithmetic, the supervisor has two specific responsibilities. The first of these responsibilities, and seemingly the one of foremost significance to the success of any efforts to promote the improvement of instruction in arithmetic, is the formulation in the mind of the supervisor of an adequate understanding of the subject itself and an understanding of clearly defined goals for instruction. The degree to which the supervisor accepts this first responsibility will determine largely his success in carrying forward the second obligation. Unless the supervisor unreservedly accepts his first responsibility, he cannot fully discharge the second, which is assisting his teachers in determining goals for instruction and in planning the program of learning activities appropriate to such goals. Thus, the supervisor will necessarily find himself concerned with several tasks. He must make available to his teachers detailed descriptions of effective teaching procedures, and he must assist them in studying and in selecting procedures appropriate to the instructional program. Also, the supervisor must be able to assist his teachers in developing experimentally units of instruction which seem to be compatible with predetermined goals. Finally, the supervisor should be able to assist his teachers in formulating adequate techniques for evaluating the effectiveness of the teaching procedures employed in the program of instruction.

Selecting Teaching Procedures

The view of the subject of arithmetic and the goals of instruction which the supervisor and his teachers develop will largely determine the criteria for selecting teaching procedures which will become a part of the instructional program. However, in assisting his teachers in carrying forward successfully the task of selecting and putting into effect teaching procedures, the supervisor will need to examine and to evaluate the effectiveness of teaching procedures reported in educational literature. In turn, the supervisor should endeavor to keep his teachers informed of his findings in the literature of experiments in teaching and to make such reports available for their study and trial in the classroom. The supervisor may find that bulletins containing annotated bibliographies and reviews of reported classroom experiments are two direct means of guiding his teachers in their study and use in the classroom of recommended teaching procedures.

In addition to the services that have been mentioned, the supervisor will usually find that interest in the in-service program for improvement of instruction will be greatly enhanced if he will make it possible for his teachers to share with each other their experiences with teaching procedures which they consider successful. To make such sharing of experiences a reality, the supervisor will probably need to be able to provide for the reproduction and distribution of reports of his teachers' experiments.

The use of the classroom demonstration as a technique for illustrating specific learning activities peculiar to particular teaching procedures will usually provide the supervisor another medium for promoting the sharing among his teachers of successful teaching experiences. Also, the supervisor will likely find the use of the demonstration as a supervisory technique an avenue which provides numerous opportunities for him to illustrate learning activities which he observed in progress in classrooms or found reported in educational literature.

Assisting Teachers in Organizing Experimental Teaching Procedures

If a program of in-service education is to be distinctive, it must be designed so as to provide opportunities for teachers to engage in experiments in teaching. Experimentation in the classroom tends to result in contributions to in-service education which have been proved significant by practical trial. Through the conduct of experiments in teaching, teachers find themselves in possession of procedures which provide day-by-day observations of how pupils learn when learning activities vary. Conducting experiments in teaching also provides teachers with those techniques necessary in evaluating adequately the effectiveness of the research of others.

The supervisor should recognize that if teachers are to organize and conduct experiments in teaching, they will need encouragement and guidance. Teachers frequently feel that their efforts in designing and putting into effect teaching procedures are not to be considered as research in teaching. They often indicate that the teaching situation is not conducive to the conduct of experimentation. Teachers frequently report that even though a particular teaching procedure when employed in a given classroom proved to be effective, it may prove to be of little or no value to other teachers. In brief, teachers too often are inclined to consider the results of their own efforts in research as being without significance. It is at this point that the supervisor will find an opportunity to encourage his teachers in recognizing that the primary value of research in the classroom is to determine the effectiveness of a set of learning activities being employed experimentally with a particular group of pupils. Also by en-

couraging teachers to repeat experiments in teaching and by urging other teachers to give practical trial to relatively untried sets of teaching procedures, the supervisor and his teachers will be able to develop a body of data having value to teachers everywhere.

In-service efforts to improve instruction should provide occasions for the supervisor and teachers to work co-operatively in planning and conducting experiments in teaching. In such an instructional program the supervisor will help to discover learning situations in the classroom which will be worthy of planned experimentation. When these learning activities are recognized, the supervisor and teachers involved will find it advantageous to the conduct of the experiment to plan together the details of the experimentation. Usually such efforts will assure a more carefully planned and conducted project.

In conducting experimentation co-operatively, there is the possibility of creating a situation in which a number of individuals with a common interest will exercise a critical attitude toward the experiment. There is also the possibility that the evaluation of the experiment may include not only an appraisal of the end result but also a consideration of the effectiveness of its several progressive stages.

Assisting Teachers in Evaluating the Effectiveness of Efforts to Improve Instruction

Merely including provisions for measuring the effectiveness of the teaching procedures being employed does not guarantee the appropriateness of the techniques of appraisal. Whether the appraisal is appropriate for the instructional procedures being appraised depends largely upon the conception of evaluation and instruction in arithmetic held by those who organize and direct the instructional activities.

If the efforts to improve instruction in arithmetic are based upon the assumption that the goals of instruction are the mastery of computational skills and the attaining of a high degree of proficiency in "problem-solving," then there seems to be the necessity for few instruments of evaluation. Various pencil-and-paper tests will usually be appropriate and sufficient to appraise the effectiveness of such an instructional program. However, if the efforts to improve instruction include, in addition to the goals described above, the assisting of pupils in developing interrelated number ideas based upon the study of the number system and the use of the ideas in the study of familiar number situations, then the process of evaluating the effectiveness of the instruction obviously will demand the employment of instruments other than formal pencil-and-paper tests.

When the goals of instruction in arithmetic are viewed as being limited to little more than the acquiring of mastery in the computational skills, the evaluation of the effectiveness of the instruction is considered a task

which needs to be done only at stated intervals. These intervals too often are not determined in accordance with the reactions of the pupils to the stages or phases of learning involved in the development of a particular set of number ideas. Instead, the periods for measuring the effectiveness of teaching procedures are frequently determined by the length of teaching units, or more often in terms of how often it is necessary, in accordance with administrative policy, to report the progress of pupils to their parents. In brief, the implication which such practices in evaluation contain is that evaluating pupils' progress is a task unrelated to instruction.

When instruction in arithmetic includes goals other than achieving skill in computation and problem-solving, the supervisor and teachers should develop techniques for evaluating the effectiveness of their efforts as they determine the learning activities for improving the teaching program. In fact, they will find it essential to consider evaluation as a continuous process which cannot be viewed apart from the learning activities of the instructional program. In emphasizing the necessity for viewing instruction and evaluation in arithmetic as interdependent, Brownell[2] offers the following comment:

> Instruction and evaluation go hand in hand. As teachers develop new insights into learning—its difficulties, its stages or phases of development, the basic understandings required for each advanced step in learning—as teachers acquire these insights, they will employ them in improved evaluation. And as they correct or modify their evaluations and devise procedures which are more comprehensive and more penetrating, they should come upon new data of great significance for the better guidance of learning. Viewed thus, instruction and evaluation are inseparable and mutually interdependent.

Such a view of evaluation and instruction as here described will make it necessary for the supervisor and his teachers to include in the teaching program not only various types of formal tests but also such techniques of evaluation as observations, individual interviews, and pupils' reports of their reactions to particular learning activities.

Observations. The chief value of observations to programs of evaluation in arithmetic lies in the fact that this technique, as well as interviews and pupils' reports, may be employed in studying the successes and difficulties encountered by pupils as they participate at each level or stage of learning. The significance of reactions of pupils to learning situations cannot be understood unless teachers continuously observe pupils at work at each level of learning. In other words, to be able to recognize all reactions

[2] William A. Brownell, "The Evaluation of Learning in Arithmetic," *Arithmetic in General Education*, pp. 228–29. Sixteenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1941.

of pupils to a learning activity, and to evaluate these adequately, teachers must observe pupils as they participate in all phases of a learning activity.

The supervisor should encourage his teachers to recognize that continuous observation of pupils includes giving attention to the informal everyday happenings of the classroom. It is important to recognize that the analysis of written work and work habits of pupils is a form of observation. Thus, to say that observations should be conducted continuously should encourage teachers to continue using many of the observational techniques which they have been putting into effect in the classroom.

In conducting observations, the supervisor should assist his teachers in being as objective as possible. The planning and conducting of observations should be an attempt to view reactions of pupils in terms of the nature of the learning process. If the teachers and supervisor plan co-operatively to make observations, there is the possibility of greatly increasing objectivity. Thus, teachers and supervisor should have frequent opportunities to observe pupils at work and to make comparisons of reactions observed.

Interviews. The supervisor will usually find that he can be of assistance to his teachers in planning and conducting individual interviews. The interview, when directed carefully, will provide teachers with a continuous record of crucial steps or phases of various learning situations in an instructional activity. Teachers must recognize that the interview must be planned for the individual being interviewed. The questions must not be stereotyped or the interview will be useless in discovering pupils' reactions to particular problems in a learning situation.

Again, it is appropriate to mention the fact that the supervisor should help his teachers realize that interviews may be informal as well as formal. He should encourage his teachers to view the informal daily discussions and questioning carried on by pupils and teachers as an interview. As informal daily discussions represent a most natural form of the interview, teachers and supervisor should endeavor constantly to improve their use of this activity.

Pupils' Reports. In having pupils present reports of their reactions to a particular learning activity, teachers and supervisor will find that they are employing a technique of evaluation which will provide data valuable in comparing results obtained from observations and interviews. When pupils independently report their reactions to significant learning situations, they are in effect exhibiting the understandings and knowledge which they brought to or have gained from the situation.

Teachers should recognize certain guideposts in the use of pupils' re-

ports. These reports may be formal or informal, and when pupils casually describe their reactions to a learning activity without being urged to do so, they are making reports. Such reports of pupils should never be turned aside as being of little importance. When pupils offer reports of their reactions, they should be permitted to tell all details without having others inject anything into the report. To be of value in furthering the progress of pupils, the reports must fully reflect the pupils' reactions.

Formal Tests. The use of formal tests cannot be neglected in a program of evaluation. Teacher-prepared and commercial tests are valuable in evaluating pupil progress. However, if evaluation and instruction are viewed as inseparable in practice as well as theory, the supervisor and his teachers will need to determine how many and what types of tests are to be employed in each phase of the teaching program. When and how often achievement, diagnostic, inventory, and survey tests are to be used is a question which the supervisor and his teachers will necessarily determine in view of the objectives of the teaching program.

Meeting Local Needs in Developing Programs for Improving Instruction in Arithmetic

Meeting the needs of local school situations in developing programs for improving instruction in arithmetic presents problems which vary widely. In organizing and putting into effect in-service programs for improving instruction, consideration must be given to the extent of the available services of the existing supervisory personnel. This consideration should include in a large urban school system not only the amount of assistance in the conduct of the project that may be expected from area or subject supervisors but should also consider the extent of the assistance which the individual school principals are able to give teachers. In smaller school systems where supervisory services of staff personnel are limited, and school principals frequently have teaching duties as well as administrative and supervisory tasks, any effort to improve instruction must be planned accordingly.

Regardless of the size of the school system and the extent of the available services of the supervisory personnel, the status of the professional education of the teachers will be a significant factor in determining the organization of the in-service program for improving instruction. For example, the teachers in one school system have given much attention individually and in groups to improving instruction in arithmetic, while in another school system, the teachers have been systematically participating in an in-service program for improving instruction in reading. One group of teachers may have taken more than a beginning step in improving instruction in arithmetic. Certainly the same in-service program for

improving instruction with identical initial efforts would not be appropriate for each group of teachers. The teachers who have given much attention to improving the teaching of arithmetic seem to be prepared for an in-service program which should possibly include at the outset a variety of activities, while an initial step appropriate for teachers who have given little attention to the teaching of arithmetic would seem to be a provision for them to learn of successful efforts in improving arithmetic instruction. In any event, the decision to put into effect an in-service program for improving instruction must be the result of numerous common agreements reached by teachers, principals, and other supervisory personnel participating in the proposed project.

Illustrative Programs for Improving Instruction in Arithmetic

The discussions presented in the remainder of this chapter must be considered in part as anecdotal due to the fact that several of the supervisory programs for improving instruction in arithmetic which are described have been organized and put into effect only recently. The descriptions presented herein contain an account of a supervisory program in a large urban school system, in a rural-urban school system, and in a single school in which the principal has classroom duties as well as administrative and supervisory responsibilities.

A. Urban School System

Mr. Bernard Glantz, Chairman of the Curriculum Planning Committee for Arithmetic in the Elementary Schools of Philadelphia, has made available in a letter to the authors the following report of the in-service program for improving instruction in the elementary schools of that city:

Building the New Guide. Our in-service program began with the building of the new guide in February, 1944. A committee of principals, teachers, and special supervisors was organized at that time in the hope that the entire program of arithmetic instruction could be carefully studied and that the best thinking of the entire system would be involved in the creation of a new guide for teachers. Questionnaires were used at times, and publications that were developed by this committee were built upon suggestions that were sent in from the entire teaching personnel. The central committee carefully edited descriptions that were submitted and incorporated them in publications that were subsequently issued. The central committee investigated the literature in the field of arithmetic and issued statements on suggested readings to the entire instructional staff. Consultants were invited to speak to groups of teachers in the various districts that composed our school system. Thus, the groundwork was laid for a fuller understanding of the program that was to follow. We were particularly concerned that, as far as possible, the entire instructional staff grow together in this sub-

ject so that when the final guide were issued it would represent a summary of the thinking of the entire teaching body.

Several bulletins were issued relating to understanding of number concepts, the place of drill in the instructional program, etc. A tentative guide was issued in 1945 as an intermediate publication in the on-going program that was then under way. This tentative guide was carefully studied throughout the city. Suggestions were made, and these served as a basis for the development of the more permanent guide that is now in use. When the new guide was issued in 1946 many teachers were prepared to receive it, since they were not only aware of what was coming but had had the opportunity to experiment with the many devices, techniques, and methods that were finally suggested.

Collaborating Teachers. The collaborating-teacher technique provided another opportunity for the in-service training of teachers during this entire period. For this purpose teachers were selected who had exhibited marked skill in modern techniques in arithmetic instruction. One collaborator was assigned to each district, and his job was to work co-operatively, as requests came in, with individual classroom teachers, with complete faculties in specific schools, or with an entire district of schools. The collaborating teacher acted as a consultant to individuals who were concerned with this program. He also gathered ideas of new practices as he traveled about from school to school and brought suggestions from one person to another. In many instances he demonstrated techniques as he went from place to place. The collaborating teacher is not a supervisor. He is exactly what the word implies. He collaborates with the classroom teacher on a co-operative basis, and, as a result, better practices have been developed.

In-service Courses. Courses have been offered by the central arithmetic committee in order that teachers may be brought up to date on modern techniques and processes. Some of these were scheduled on a city-wide basis and others on a district-wide basis in order to meet the needs of the various community groups. The attendance at the beginning was quite large. In recent years the numbers have grown smaller because so many people have already taken these courses and because the new guide is now in the hands of the teachers.

Film Slides. Four sets of film slides have been developed by the central arithmetic committee so that principals may use these with teachers in faculty meetings. A great number of others will be developed from time to time so that the specific difficulties that concern teachers may be properly illustrated and discussed by them. The titles of these films are: "Beginning the study of Number"; "The Study of 6"; "The Study of Fractions"; "The Study of 10 as a Group."

A recorded disk accompanies each set of slides so that teachers receive an auditory impression as well as a visual impression in considering the topics above.

The Meaning-Inventory Program. Philadelphia has developed a series of tests involving the meaning of number. Three such tests are now available out of a possible thirty that will be constructed. One deals with vocabulary, another with coins, and a third with fractions. Teachers are encouraged to suggest specific items that might be incorporated into these meaning tests so that the

entire venture is a co-operative one and certainly provides opportunity for teacher growth.

Concrete Materials. Seven kits of concrete materials are being prepared for use in the seven districts so that teachers and principals may be thoroughly acquainted with the kinds of materials they can order for use in classrooms. Since concrete materials are so important in our arithmetic instruction, it was felt that these sample kits might be of service to the teachers not only in knowing what to order but how to use them as well.

Radio Programs. A series of demonstration lessons entitled "The Classroom of the Air" is presented each year by radio, and pupils as well as teachers are prepared in advance to listen to them. These illustrate a variety of techniques to be used in various topics as suggested by the arithmetic guide. It is not unlikely that "The Classroom of the Air" will find its way into the television field in the near future.

Clinical Processes. The central committee has assigned to a relatively small group of teachers and principals the task of determining whether or not it would be desirable to develop an arithmetic clinic to determine the causes of inefficiency in arithmetic on the part of pupils. This study is taking place at the present time; no significant results are as yet available.

Supplementary Publications. Our arithmetic guide will never be in final form. We are committed in Philadelphia to an on-going program and hope in the future to make changes in the guide every few years. These changes, however, will not be so revolutionary that our teachers will be faced abruptly with an entirely new program. We hope to accomplish this task in the following way. Supplementary publications will be issued from time to time clarifying statements that are made in the guide and presenting new views on techniques already suggested. From time to time these supplementary publications will be incorporated into revisions of the arithmetic guide.

Textbooks and Supplies. Our textbook and supply committees consist of principals and teachers. In examining the supplies that are offered for listing and books to be made available to teachers, the committee has a grand opportunity for growing in ideas relating to the field of arithmetic. Suggestions of this committee are employed in the final listing of both textbooks and supplies.

B. Rural-Urban School System

During the school year 1939–40, Wilburn,[3] a supervisor in Berkeley County, West Virginia, Schools, in co-operation with fifteen teachers in one- and two-teacher schools developed a teaching program which resulted in having pupils in Grade I teach themselves, largely by their own efforts, the forty-five addition and subtraction facts with sums and minuends of ten or less. The supervisory program consisted largely of group and individual conferences with teachers, the preparation of a series of

[3] D. Banks Wilburn, "A Method of Self-instruction for Learning the Easier Addition and Subtraction Combinations in Grade I," *Elementary School Journal,* XLII (January, 1942), 371–80.

bulletins describing the several teaching procedures to be employed in the classrooms, demonstrations, and classroom visitation.

The group conferences with the teachers who participated in the experiment occurred monthly. The first two conferences of the school year were given over to discussions of the objectives of the teaching of arithmetic and to specific understandings necessary to the conduct of the teaching experiment. The subsequent conferences were concerned with specific discussions of the learning activities which the teachers were employing in the classrooms. Following the conferences, summaries of the discussions were prepared and distributed to the teachers.

Bulletins were prepared in which such subjects as, "Methods of Study To Be Used in Learning To Count," and "Methods of Studying Groups," were discussed.

With the consent of the teachers, the progress of the experiment was observed as it was in operation in the classrooms. Following the observations, conferences were held with the teachers. Particular learning activities were discussed with the teachers in order to clarify any phase of the work for either pupils or teachers.

In many cases, the observations and conferences with the teachers resulted in requests for classroom demonstrations of certain phases of the instructional procedures. In almost every classroom in which the experiment was being conducted, the teachers requested demonstrations of the initial learning activity of each new unit of instruction.

During the next three school years, Wilburn[4] and the fifteen teachers who participated in the program described above were joined by the teachers of six elementary schools, having four or more teachers each, in developing and administering a teaching program designed for the purpose of having pupils largely teach themselves a content of arithmetic appropriate for Grades I, II, and III. The supervisory program of this teaching experiment was, with one exception, very similar in organization to the program presented in the preceding paragraphs. The difference in the organization of the two supervisory programs was due mostly to the assistance given the teachers participating in the program for the first time by the teachers who participated in the teaching experiment in Grade I. Also, the six elementary-school principals gave valuable supervisory assistance to the conduct of the teaching experiment. The fifteen teachers in the rural schools gave much aid in the discussions of the group conferences and in conducting demonstrations in the classrooms, and the principals assumed such responsibilities as conferring with teachers, observ-

[4] D. Banks Wilburn, "A Particular Program of Instruction in Arithmetic in Grades I, II, and III." Unpublished Doctor's Dissertation, George Washington University, 1945.

ing pupils at work, and assisting teachers in initiating teaching procedures in the classroom.

C. A Small Elementary School

Mr. M. V. Givens, Principal of the Jayenne Elementary School, Fairmont, West Virginia, whose responsibilities include teaching in the classroom as well as administrative and supervisory duties, has reported in personal correspondence the procedures which he and his teachers are employing in improving instruction in arithmetic. In the report which follows, Mr. Givens has presented the several procedures being applied and his views regarding their significance for the improvement of instruction in arithmetic.

Promoting Teacher Growth

The in-service growth of teachers may be stimulated through a variety of procedures. A few important ones follow.

Subject-Matter Conferences. Since the principal is the co-ordinator of the learning activities of his school, he finds it expedient to call his teachers together from time to time for subject-matter conferences. These conferences provide opportunities to: (*a*) plan for continuity from grade to grade (or from classroom to classroom); (*b*) exchange ideas on a variety of topics related to the subject; (*c*) point out weaknesses in the program; (*d*) plan new procedures; and, (*e*) review important source materials. Teachers generally profit from these experiences.

Encouraging Initiative. Teacher initiative and success need to be acclaimed by the principal as supervisor. While it is necessary to have common objectives and a consistent program in the subject of arithmetic, we can avoid the dulling effect of "cut and dried" procedures or stereotyped lesson plans. The enthusiasm of the teacher and the novel approach often mean the difference between success and failure in the classroom. We admit that individual differences among teachers are as great as the differences among children. The principal's recognition of a teacher's originality in instruction and of her efforts serve as incentives to teachers to seek still better methods and to strive for better results.

Citing Special Articles and Books. The principal can offer leadership by citing or reviewing important articles and books on the subject. Reports of experimentation and research should be made available to teachers. These reports should be analyzed and evaluated critically.

Demonstrations. Teachers may profit from the demonstration of certain procedures and devices used by others in their teaching. When requests are made for such demonstrations, the principal, as supervisor, should see that they are provided. No doubt, he may desire to explain or demonstrate some of his own methods.

Providing Materials. All necessary materials for effective teaching should be available and put to use. There are many visual aids that can be employed with profit in the arithmetic class. The use of many of the common measuring instru-

ments should not be neglected. Construction materials of various kinds have value in building number concepts. Abstract ideas can be made meaningful when it is shown that they are derived from realities.

Special Methods. A great need on the part of many teachers is an understanding of methods which they may teach pupils to help them "figure things out for themselves." Such methods have been referred to as methods of self-instruction. Ample proof exists that such methods can be taught to and used by relatively young children without undue effort. In fact, we know that pupils frequently devise methods of their own for this purpose. But oftentimes their methods are faulty and need to be canceled or improved. The value of good methods of self-instruction is too great to be left to chance. Under the stimulus of a self-instruction plan, pupils gain in independence and self-reliance—two values to be desired in any program. The supervising principal should help the teacher develop methods of self-instruction.

Guiding Experimentation. A teacher needs to engage in experimentation. He can evaluate the research of others more intelligently if he has carried out an experiment of his own. And, it offers opportunities to observe how pupils actually learn. The supervising principal may serve in an advisory capacity and offer the assistance and encouragement necessary to complete such a project.

The Supervising Principal in the Classroom

Studying the Needs of Pupils. The supervisor will want to see the program in action. This calls for firsthand observation. He needs to consider the effects of the program as reflected by the reactions and attitudes of the pupils. Among other things, he will study the needs of pupils. He will observe their work habits; he will note their difficulties and special weaknesses; and, he will be interested in the methods which they use in seeking solutions to their problems.

Assisting the Teacher. The pupils will welcome the supervisor. He helps them with their work by assisting the teacher in caring for individual needs. Through good questions, he leads them to employ problem-solving methods in their work, and he stimulates insightful learning. He directs them in self-instruction.

Conferring with the Teacher. As a result of his classroom visit, the supervisor will be able to discuss with the teacher the teaching and learning activities which he has witnessed. Special strengths and weaknesses will be considered, and recommendations may be in order. The teacher will welcome the opportunity to discuss problems with the supervisor because he is a co-worker who is interested in the success of teachers as well as pupils.

Evaluating the Program

As a consequence of his classroom visits, the supervising principal has obtained much information that will aid him in an evaluation of the program. He has noted the attitudes of the pupils; he has witnessed the progress from grade to grade and from year to year; he has attempted to evaluate the ability of pupils to apply their knowledge in the solution of their problems. And, through individual and group conferences, he has gained the benefit of the judgment of teachers.

The supervisor may, however, desire more objective evidence of the effects of the program. He will devise, and assist his teachers in devising, a number of objective tests to help measure the results of instruction. The purpose of many of these tests will be to measure growth in the ability to generalize, to see relationships in the number system, and to exercise insight.

These accounts of the organization of in-service programs show some differences in techniques and methods. Such differences are, of course, necessary to meet conditions in schools in various kinds of communities. The accounts exhibit fundamental similarities, however. They all are based on a conception of supervision which stresses participation of staff members, the attempt to solve important instructional problems through a co-operative approach, and the necessity for skilful and understanding leadership.

Summary

It has been the purpose of this chapter to indicate some principles and techniques for organizing a program for the improvement of the teaching of arithmetic. The specific proposals which have been made in the latter part of the chapter are part of a larger conception, which maintains that the role of supervision is one of improving instruction through staff participation and professional leadership. The point of view expressed throughout the chapter is that the traditional supervisory activities of prescription of content and method and inspection of the work of teachers are inferior to the modern conception of supervision as mutual study of important problems by the school staff under able leadership.

The chapter has also pointed out the importance of a sound knowledge of arithmetic on the part of teachers who are expected to organize the number program for children. It has been suggested that one fruitful approach to this problem lies in a historical study of the number system by teachers.

In addition to such a study of the content of arithmetic, it has been suggested that teachers will also profit from a study of the relation of the psychology of learning to the arithmetic program. In this connection it has been suggested that an academic approach to the psychology of learning be avoided. The idea should be to use principles and data from psychology to aid in the solution of instructional problems.

Finally, the chapter has presented concrete examples of programs which have been designed to contribute to in-service development of teachers of arithmetic. Examples have been presented which are drawn from schools in widely different kinds of communities to show how supervisory programs may be adapted to local needs.

There are many perplexing problems connected with the improvement of the arithmetic program. In any given school only a sustained attack in which all members of a staff participate under competent leadership is likely to be sufficient.

CHAPTER XIV
THE SOCIAL POINT OF VIEW IN ARITHMETIC

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Arithmetic Evolves from the Emerging Needs of Human Society

The body of doctrine which we now know as arithmetic and a portion of which we teach to young children began as a response to human needs.

The cave man, naming his possessions because he could not count them and later dealing with representative groups by a crude one-to-one correspondence, was meeting his needs with such arithmetic as he had at his disposal. To him, one-to-one correspondence was not a theory but an empirical fact; something not of remote meaning but of immediate value. The shepherd, telling his flocks with sticks and stones and fingers, or counting his sheep as far as he could go with names derived from natural objects, was likewise using such arithmetic as he knew in order to meet his problems. Giving names, tallying, comparing and combining actual groups, and counting—these, it is clear, must have been evolved to serve human purposes. Primitive men can hardly be said to have invented or discovered their arithmetic; they lived it.

The social value of arithmetic, that is, its human serviceability, is no less evident in the historical period. Units of measure—originally, a stone or a vessel of water for weight, a well-marked part of the human body for length, a gourd or a shell for capacity—eventually became uniform over considerable areas under names almost forgotten, such as the *mina* and the *talent*, the *palm* and the *cubit*, the *hin* and the *amphora*. Fractions were devised and gradually assimilated into the realm of number, necessitating new rules of operation. Zero was invented and led at once to positional notation for whole numbers and later to the same notation for fractions. These achievements and many more of similar nature were triumphs of the human spirit. They responded to the needs of human society as that society became more complex. They were not merely discoveries like gold found in the hills or a new star located in the firmament; they were growths.

They were also building blocks, hewn by ancient workmen, yet destined to be used later in erecting the Temple of Mathematics. The men

who shaped these stones were widely scattered—a few in Egypt; a few more in far-off India; and yet others in Babylon, China, and the Near East. We have no reason to suppose that they thought they were contributing to the building of an organized system. Facing simply and even naively the manifestations of Nature, they sought to come to terms with her.

Yet they *did* contribute to the building of the Temple. The stones they had fashioned were not rejected, for they were found to fit together, both those which had been contrived in the same locality and those which had been shaped in distant places. These workmen confronted Nature and worked with her. Their product, therefore, though scattered in time and space, partook of the unity of Nature. Accordingly, the work of these ancient craftsmen entered inevitably into an organized and interrelated system, every part of which has been found to be true in the sense that it is without contradiction.

In this light, let us consider further the three ideas above mentioned, namely, units of measure, fractions, and zero.

Units of Measure

To measures of continuous magnitudes were linked counting numbers and fractions, that is, numbers developed for the quite different purposes of describing the plurality and defining the parts of discrete objects. Then, with the increasing exactions of a more complex society, came new concepts of approximation, of precision, and of error, along with the devising of more adequate numerical expressions for these ideas. It also became evident that units of the same kind—say, units of length—would be more useful if they could be combined so that each unit of a higher order would be an exact multiple of the units of lower order in the same series. Hence arose tables of measure, such as: 4 digits = 1 palm; 3 palms = 1 span; 2 spans = 1 cubit.

Many steps had to be taken before such a result was achieved. Indeed, in the English system it has not yet been fully achieved. We still have to deal with the fact that $5\frac{1}{2}$ yards make a rod and with the related fact that $30\frac{1}{4}$ square yards make a square rod. At first only pairs of measures were integrally related. For example, in the ancient Hebrew system of dry measure, four *logs* made a *cab* and 10 *omers* made an *ephah*, but there was no integral relation between the *cab* and the *omer*.

Measurement systems were further improved by establishing relationships between units of different kinds. Time and length were related when the length of a shadow told the hour of the day. Much later, time and length were more scientifically related by the so-called second's pendulum. A relation among measures of length, capacity, and weight is evi-

dent in the Babylonian definition of the talent as the weight of a cubic foot of water.

Sometimes the drive for an articulated system was stronger than well-established units. The cubit and the foot, both natural units and both widely used, were numerically antagonistic. The cubit was not a multiple of the foot; it was $1\frac{1}{2}$ feet. So the double cubit, already used by builders, was widely adopted. In English it was called an ulna or an ell, and finally a yard. Hence the familiar fact, "3 ft. = 1 yd."

In such ways a whole new order or "side" of arithmetic has been created. We may call it measurement arithmetic. Its high social value is attested by John Quincy Adams when, in his famous report, he has this to say about weights and measures:

> They enter into the economical arrangements and daily concerns of every family. They are necessary to every occupation of human industry; to the distribution and security of every species of property; to every transaction of trade and commerce; to the labors of the husbandman; to the ingenuity of the artificer; to the studies of the philosopher; to the researches of the antiquarian; to the navigation of the mariner, and the marches of the soldier; to all the exchanges of peace, and all the operations of war. The knowledge of them, as in established use, is among the first elements of education, and is often learnt by those who learn nothing else, not even to read and write.

Fractions

Fractions, originally thought of as parts of a unit and therefore as less than one, were extended in the historical period to mean any rational number, large or small, whole or fractional. Furthermore, a fraction, which was at first generally regarded merely as a "broken number," came to be understood likewise as a way of expressing the division of one number by another, or, more generally, the relation of one number to another. This idea of the meaning of a fraction played a part in developing such fields as percentage, ratio and proportion, permutations, combinations, and probabilities. Moreover, since the related numbers need not be integers, compound fractions arose. By controlling the denominators of fractions according to the powers of ten, decimal fractions came upon the scene. The fraction system was also extended to include continued fractions and partial fractions.

Zero

Zero made possible the development of directed numbers. It became the point of origin, not only of real numbers, but also of imaginaries. By making positional notation possible, it introduced entirely new and better ways of computing—the famous algorisms—and revolutionized opertional thinking with numbers. Zero banished the abacus and thus released the computer from dependence upon a machine. It led to the conception

of a decimal number as the sum of as many terms as it has digits other than zero.

Accretion and Accrual

The three ideas we have discussed—namely, measures, fractions, and zero—seem to indicate rather clearly that arithmetic has had a cumulative growth from prehistoric times and that this growth is of two kinds.

First, there have been certain primary discoveries. These are the essential basic ideas. Most of them were gained by trial and error. In reference to the body of knowledge already set up before their arrival, they are true accretions. Only to a minimum degree are they suggested by arithmetical ideas already current. Primarily they are responses to human needs. They come from without; and practically all of them date from prehistoric or early historical times. The last great accretion was the discovery of logarithms.

Second, in addition to accretions motivated by human need, the cumulative development of arithmetic has been due to inner growth, to accruals resulting from the deposit of original ideas. These increments to the original stock are due in the main to mathematical thought and research. Each such increment has been based upon the reliable findings of an earlier period. It is true that mathematical research sometimes finds errors or shortcomings, but even this corrective work is part of a great forward movement.

Advances based upon previous well-tried ideas—advances which are in general mathematically motivated—may belong to the realm of so-called pure mathematics. Again and again, however, the distinction between pure and applied mathematics has proved to be a false distinction. For example, prime-number theorems, binary numbers, and continued fractions may once have seemed too theoretical for practical use. Yet this has not turned out to be the case. A glance into an engineers' handbook will show the surprising extent to which prime numbers are used in practical engineering. Some of the marvelous computing machines lately made available are constructed in accordance with the binary number system. And modern airplane design is said to make frequent use of continued fractions. The dichotomy in virtue of which arithmetic is supposed to be either pure or applied is unhelpful.

The Effects of Arithmetic on Social Institutions

A consideration of the sociology of arithmetic is surely incomplete if it fails to take into account the impact of the subject upon the organization of society. Astronomy had its origin in Babylonia; surveying, in Egypt. Without arithmetic, the stargazers of Mesopotamia would have remained mere shepherds stupidly wondering at celestial phenomena.

Without arithmetic, the farmers of the Nile would have lost each year the boundaries of their fields and would have continued, like their ancestors, to settle their disputes by fighting and plundering. Priests at once became the monopolists and the preservers of learning. Moreover, they contributed to learning. It was doubtless as a result of temple research that various number systems arose.

By means of these number systems, records were kept. Calculation—hardly worth while if its results must be held in mind for long periods and communicated (and disputed) by word of mouth—was more and more undertaken as number systems improved. Calendars were devised as a special kind of numerical record, and chronology had its beginnings. The idea of money developed, and coins were minted. Taxes were levied, and strong governments arose. Business and trade took the place of barter and pillage. Antiquaries delving among the ruins of ancient Babylon have found promisory notes, mortgages, and letters of credit on clay tablets.

Each of the obvious influences of arithmetic which we note in ancient times has been heightened in our own day. Our astronomers make far better predictions, their conception of the universe is more convincing, their theory of planetary development is more probable, their navigation tables are incomparably more serviceable, and their calendars are more accurate. Our surveyors, likewise, have at their service a better mathematics and therefore do a better job. With better instruments and a better arithmetic they run courses and distances with far greater accuracy so that we may possess our land in security and guarantee it to our assigns. With a better arithmetic they have developed map-making into a science—that of the cartographer. With a better arithmetic they have carried their surveys across broad rivers and over high mountains and through deep forests until these surveys have covered vast areas of the earth and, ceasing to be local, have become geodetic.

The Effect of Social Institutions on Arithmetic

Arithmetic in its relation to social institutions is not only a cause; it is also an effect. Institutions have influenced arithmetic.

First, however, let us observe that the term *arithmetic* has a dual meaning. If I say of a person, "He is good in arithmetic," I mean that he performs the operations of arithmetic easily and accurately; I think of arithmetic as a process. On the other hand, I may speak of arithmetic as a subject. I then have in mind a certain structure consisting of many parts—a structure which has its terms, its postulates, and its logic. Clearly, arithmetic has a twofold meaning just as some of its own terms and symbols may have double meanings. For example, division, in spite of the fact that as an operation it is a single process, has in reality two dis-

tinctly different meanings. Again, the symbol $\frac{2}{3}$ may mean, on the one hand, the relation of 2 to 3—that is, $2 \div 3$—while, on the other hand, it may be a number.

Out of this dualistic meaning of arithmetic, a certain interesting fact arises. When emphasis is placed upon operations, arithmetic clearly influences society, whether in primitive or ancient or modern times. The reader will note that the illustrations of the influence of arithmetic upon social institutions, as these illustrations were presented in the last section, were examples of the social applications of arithmetical operations. Arithmetic so conceived lends strong support to certain social ideals and objectives, such as precision, punctuality, order, safety, health, and prosperity. On the other hand, as a structure—that is, as a subject—arithmetic has been in the past and is today profoundly affected by the prevailing type of society. In Greek and Roman times the term *arithmetic* had no operational meaning. This honored title was bestowed upon a body of doctrine highly speculative in character, futile, and sometimes false in its distinctions. It is not clear that calculating was taught either at Athens or at Rome. The chief uses for skill in reckoning were in activities which were not performed, or at least were as far as possible avoided, by the elite.

If we seek for reasons why arithmetic in the Greco-Roman period was of this nature, we shall find the most important reason within the structure of society itself. It was a slave society. Physical labor was despised. The production of goods for general consumption was in the hands of the lower classes. Only the intellectual, the artistic, and the recreational were proper fields for the Greek or Roman citizen. Except in emergencies, even the military field was somewhat beneath one's dignity, for soldiers could be hired. And, as this type of society worked out its degenerative effects, mercenaries fought its battles.

In such a society, arithmetic as a body of doctrine rather than a series of more or less unrelated skills was the only arithmetic recognized. Its character was the product of the society in which it prevailed. Philosophers speculated about the properties of number. A cult of number arose, and number mysticism engaged the attention of thinkers. Many of their ideas were fantastic. Arithmetic was not expected to serve the community. It was not operative. What we now regard as the arithmetic of operations bore a different name in ancient Greece; it was called logistic.

The critical event which rescued arithmetic from the Platonic school was the triumph of a nonwestern type of arithmetic centering upon a positional notation to the base ten. Moreover, it is not difficult to show that the struggle—which for centuries preceded this triumph—was a struggle of rival cultures. It is of course important, though not determina-

tive, that the relatively weak Roman number system rather than the stronger Greek system was then dominant in the West. But there is definite contemporary evidence as to the general hatred of western culture in non-Christian areas. This hostility was one of the chief causes of the easy military victories of Islam. It accounts, in some measure, for the mathematical school at Bagdad, a school which looked elsewhere than to the West for its lessons.

It is customary to regard the discovery of zero as "a stroke of genius" or as "a gift from blind chance." This suggests that zero had no antecedents and, in particular, that its discovery owed nothing to a social background. Yet, why was it the Hindus who made zero a going concern?[1] Why, for example, did the Greeks, with their remarkable cleverness in mathematics, fail to attain it? The Hindus did it because they were Hindus. It has been well said that if there was any invention for which the Hindus, by all their philosophy and religion, were well fitted it was the invention of zero. "This making of nothingness the crux of a tremendous achievement was a step in complete harmony with the genius of the Hindu."[2]

Although the Hindu-Arabic system was technically superior to existing systems, its spread over western Europe was due in large measure to social conditions. First, the communities of Europe were too weak, ignorant, and divided to defend their number system even if it had been a better one.

Second, the establishment of the magnificent school at Bagdad (in a way, a school of protest against the West) offered an opportunity to Al-Khowarazmi and other Arabian scholars to write on the new system of numbers and to state its case with authority.

Third, the aggressiveness of Islam enabled the Arabs vigorously to perform their office of transmitting the Hindu number system to western Europe via Spain and Sicily, thus explaining, if not justifying, the addition of the term "Arabic" to the term "Hindu" in naming the system.

Fourth, social conditions accounted for a considerable amount of communication between East and West. Caravans and galleys brought knowledge of the new system. Occasional ambassadors were sent to east-

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[1] The attainment of the Hindus is phrased this way out of deference to those who are unconvinced that the Hindus invented zero. The question is largely a semantic one. For the present purpose it is immaterial whether the Hindus invented zero in the loose or restricted sense. They did follow it up with the "necessary conclusions." They did write about it and they did transmit it, enriched by their own thought, to the Arabs.

[2] David Eugene Smith and Louis Charles Karpinski, *The Hindo-Arabic Numerals*, p. 43. Boston: Ginn & Co., 1911.

ern potentates and returned with tales to tell. Pilgrims plodded their weary way to eastern shrines and returned to be venerated. Pillaging, no less than trade, brought the peoples of the East and West into contact. The Crusdades must also have played an important part in acquainting the West with the East. Intercourse between the East and the West justifies the belief that the Hindu numerals "must, of necessity, have been known to many traders in a country like Italy at least as early as the ninth century."[3]

It was not, however, until the beginning of the thirteenth century that a competent mathematician (Leonardo of Pisa) wrote an account of the new number system and the way to compute with it. Yet three more centuries were to pass before the adoption of the new numerals can properly be said to have come to pass. This, too, was due to social conditions. For society, in its effect upon the development of an institution, can retard as well as accelerate. Conservatives were solidly against the new system; universities forbade its use; and the church strongly supported the Roman system. Then, too, we must consider the illiteracy of the times and the fact that, prior to the invention of printing with movable type, few books were available even for those who could read them. Chiefly, therefore, the transmission of knowledge was by word of mouth; and we should bear in mind that the Hindu-Arabic system was a written, not a spoken, symbolism.

When all is said pro and con, the one social force which, more than any other, finally assured the victory of the new numerals with their superior ways of computing was the rise of a commercial class in Europe in the sixteenth century. Arithmetics in the new genre appeared rapidly. Italy, where commerce was highly developed, led the way. Books by Tagliente and many others were amazingly popular and ran through many editions. In France, Villefranche wrote an excellent book, published in 1520. England had Robert Recorde, and Germany had Adam Riese—authors of arithmetics which were enormously popular. In the Low Countries, the most famous mathematician of this period was Simon Stevin. In his arithmetic, published in 1585, he set forth definitely, for the first time, the theory of decimal fractions. With this book the victory of the Hindu-Arabic numerals and of the algorists over the abacists was completed. Positional notation to the base 10 had not only won its way for whole numbers but had been carried to orders below the first, thereby expressing fractions and permitting calculation with them in the manner of calculating with whole numbers.

In our day we have even more massive evidence of the effect of shifts

[3] *Ibid.*, pp. 109–10.

in social needs. Just as the invention of machines fifty years ago radically changed our ideas about handwriting, so the invention of machines is changing our ideas about arithmetic. Business no longer needs the services of rapid calculators; it has them in the machines it has installed. What it needs is people who understand the meaning of the numbers which the calculating machines produce, people who draw proper inferences from numerical data, reach valid decisions, and make superior choices. Engineers, economists, and statisticians no longer "foot" long columns of figures or laboriously figure comparisons and relationships. They interpret the results of such computations. We teach computation in our schools, to be sure, and we should teach it well; but speed and accuracy as an end in itself is yielding to more humanistic objectives. Approximations, sufficiently close for the purpose, now take the place of accuracy to the last decimal, and the term "approximate arithmetic" is now heard.

Consider in this connection "The Second Report of the Commission on Postwar Plans" (*Mathematics Teacher,* May, 1945). The Commission offers a check list of twenty-eight "essentials for functional competence in mathematics." Of these, nineteen are in the field of elementary-school arithmetic; and of the nineteen, these fourteen are essentially noncomputational.

1. Has he [the pupil] the fixed habit of estimating an answer?
2. Does he have a clear understanding of ratio?
3. Is he skillful in the use of tables?
4. Does he know how to use rounded numbers?
5. Can he use averages? Can he make and interpret graphs?
6. Can he estimate, read, and construct an angle?
7. Does he know the meaning of measurement, of a standard unit, of the largest possible error, of tolerance, and of the statement, "a measure is an approximation?"
8. Can he use certain measuring devices such as an ordinary ruler, other rulers (graduated to 32ds, to 10ths of an inch, and to millimeters), compasses, protractor, graph paper, tape, calipers, micrometer?
9. Can he make a scale drawing and use a map intelligently?
10. Does he know how to use the most important metric units?
11. Does he know the meaning of a formula?
12. Does he know from memory certain widely used formulas relating to areas, volume, and interest, and to distance, rate, and time?
13. Does he have the information useful in personal affairs, home, and community—e.g., planned spending, the argument for thrift, and understanding necessary in dealing with a bank and keeping an expense account?
14. Does he have a basis for dealing intelligently with the main problems of the consumer?

In this list, the following noncomputational terms appear: *estimate* (twice), *understanding* (twice), *know how* (twice), *know the meaning* (twice), *interpret, use, make, intelligent.*

An Arithmetic Theory for the Schools of Today

Fortunately, much reconstruction is taking place in the arithmetic course—a proper step in any subject when old practices are carried into a period of changed social conditions. In such reconstruction there is abundant justification for a social theory of arithmetic. Indeed, if the historical evidence were far less convincing than it actually is, arithmetic, like any other subject in the school curriculum, would still have to prove its service to humanity. The elementary school is close to the people, and on no other ground than social service can its offerings be justified at public expense. The social aspect of arithmetic will be found to be present in every classroom; and it will profoundly affect teaching method.

By no means, however, can the curriculum fail to take account of arithmetic as a mathematical subject. Indeed, the mathematical aspects of arithmetic are the indispensable basis of its social usefulness. It is its mathematical nature which makes arithmetic what it is, the beneficent servant of man. Accordingly, its main headings as a school subject and most of its subheadings will be mathematical. We cannot escape the necessity of respecting the structure of arithmetic.

It seems reasonable that a social theory of arithmetic should have a name for the applied arithmetic which is most obviously and directly of social value. Let us call such arithmetic *significant.* By the significance of arithmetic we refer to its value, its importance, its necessity in the contemporary social order. We also mean the role it has played in science and in the development of social institutions. We mean, too, the instrument it has proved to be in ordering man's life and his environment. It will readily be seen, therefore, that the idea of significance is *functional.* It conceives of arithmetic as having something to do.

On the other hand, the term "meaning" as applied in arithmetic may properly be considered as mathematical. In using the terms "meaning" and "meaningful," we conceive of arithmetic as a closely-knit quantitative system. It has outer limits far beyond the aspects of the subject which we teach to children. The teacher should be in possession of at least a part of this greater range in order to bring to bear in the classroom the enrichment of marginal mastery. For example, the domain of number transcends the field of arithmetic. The comparatively restricted field of real numbers goes beyond the curriculum of the elementary school. Even in the realm of the so-called natural numbers the series is infinite; "there

is no last number." Thus we are confronted by the concept of infinity—even as we deal with little children.

How far, then, shall we go? There are those who tell us that we should confine our attention, for the most part, to numbers of four or five places. Yet in these days little effort is needed to show the prevalence of astronomical figures in newspapers and magazines or in any conversation that rises above the level of the weather and common gossip. Doubtless one does not need to compute *with all the digits* of millions and billions. This is where an intelligent use of "approximate arithmetic" comes to our assistance.

The teacher who emphasizes the social aspects of arithmetic does well. But, in the sense in which the term is here used, he is not teaching the meaning of arithmetic. He may, and indeed he should, use a socially significant approach, but his teaching of a given unit is not complete until the goal of mathematically meaningful ideas has been reached. These ideas should then find their application in social situations, either actual or described; for it is the application of number to the affairs of life which justifies arithmetic in the common school. Since, however, the facility with which one may apply arithmetic depends squarely upon one's apprehension of the meaning of arithmetic, it is clear that significance and meaning are indispensable to each other. We must, therefore, do two things. We must teach arithmetic as a social study, and we must teach it as mathematics. The one emphasis will exalt arithmetic as a great and beneficent human institution, the supporter of a fine human tradition. The other emphasis, the mathematical one, will lift arithmetic, even in the primary grades, from formalized symbolism to the dignity of quantitative thought.

In previous paragraphs we have seen the destructive effects of a double standard. There were the speculative arithmetic and the computational arithmetic of the Greeks. There were also, in medieval times, the Hindu-Arabic and Roman systems. In each case, the dichotomy had to be resolved before its destructive effects could be remedied.

The doctrine of meaningful arithmetic as mathematical and significant arithmetic as social must be fused into something more powerful than either of them. This fusion can take place only in the mind of the learner. The fusion is called "insight." With reference to arithmetic it is the product of the experience of mathematical truth and practical workability. Meaning and significance can be regarded as two different ideas; but to be of human worth, these two ideas must be brought together by human beings endowed with the power of synthesis. If no fusion of meaning and significance were possible, those who urge the suppression, or at least the

subordination of one to the other, would be on sound ground. As a matter of fact, however, meaning and significance are attributes external to the learner who, as he reorganizes his experience, merges them and attains insight.

Obviously, therefore, when we speak of insight we are no longer thinking of arithmetic but of the learner. Insight is a personal achievement. Without it, according to Gestalt psychological opinion, no learning takes place. In this sense it is far more important than any of the elements by means of which it may be attained. It outranks meaning and significance. It is more essential than drill or interest or effort.

Insight is important not only because it conditions learning but also because it produces an attitude favorable to the thing learned. Our fellow citizens are in sore need of a more favorable attitude toward arithmetic. Simeon Strunsky in the *New York Times* for September 29, 1946,[4] says in part:

> Sad to say, respect for arithmetic has never stood so low as it does today in this country. . . . . [People] find that arithmetic is much worse than untrustworthy. They find it a bore. At least this is the caution always addressed to writers for the press and authors of books designed for the general public. . . . .
>
> And yet the one problem which has come to transcend every other human interest today, or is described as such on every hand, is a problem of arithmetic. What is this atomic age upon which humanity has entered, if not the age of an awesome arithmetic? The atomic bomb is our first visible sample of the overwhelming arithmetic in Einstein's formula for the equivalence of matter and energy. To get the energy or force locked up in a piece of matter you simply multiply the mass by the square of the speed of light, that is all. You work out an arithmetical sum in which one step consists of multiplying 186,000 miles a second by 186,000 miles a second. The consequences are Hiroshima and Nagasaki and Bikini; otherwise arithmetic is a bore.

In the classroom, social significance is quickly and effectively invoked. Mathematical meaning, however, although ultimately more powerful as a type of learning experience, does not often lie on the surface. It is not what a teacher does now so much as it is what he did last week and last month and what his predecessors did in previous years. Meanings are not isolated facts. They exist in strands stretching from simple and concrete to complex and abstract. The strands themselves are related to each other by cross-strands. Moreover, many strands are continually combining to make fewer strands.

If we accept the current philosophy as to the value of meaningful

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[4] Quoted by William L. Schaaf, "Some Sidelights for the Arithmetic Teacher," *School Science and Mathematics*, XLVII (April, 1947), 321–22.

arithmetic, we must be prepared to begin that type of arithmetic when children enter school. We must select and co-ordinate such meanings as are likely to favor insight. We must grade these meanings according to the maturity of the child. We must build our arithmetic from the beginning with the *intention* of teaching it with meaning. We must devise ways at every level of evaluating understanding. And, as rapidly as possible, we must adopt a new program of research which will confirm, modify, replace, or reject our "informed judgment" concerning these matters.

Finally, our day-to-day guidance of the learners for whom we are responsible must reach for insight, for an illumination of spirit in which an understanding of the inherent nature of arithmetic combines with a lively sense of its instrumental value to produce personal mastery.

CHAPTER XV

NEEDED RESEARCH ON ARITHMETIC

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This chapter consists of two parts. First, there is a brief consideration of some problems of research as they relate to arithmetic; and, second, there is a group of specific proposals for research submitted by some twenty members of the profession who have an active interest in the teaching of arithmetic. The committee hopes that teachers interested in some of the particular problems that are suggested will correspond with the persons who proposed them so that all may benefit from sharing their interest in the problems. In that sense, this chapter may function as a clearinghouse for those interested in doing research in this field.

Research and the Improvement of Instruction in Arithmetic

Changing Concepts and Objectives

The most notable development in the field of arithmetic during the twenty years since the publication of the National Society's Twenty-ninth Yearbook has been a series of changes clustering around the phrase "meaningful arithmetic." One of these changes has been an emphasis on an objective of a better understanding of the number system and an increased ability in quantitative thinking. Another change has resulted in an attempt to identify those arithmetical meanings around which the content of arithmetic should be organized. Still another change has been the broadening of the base for evaluating outcomes of the teaching of arithmetic, which go much beyond the earlier objectives of computational efficiency. While the ideology of an improved program of arithmetic has been rather clearly defined, the realization of its benefits is hampered by the lack of research to match the program. For example, teachers who work for the development of meaningful outcomes are frustrated by the application of tests which measure only the computational outcomes of a drill program. Yet the improvement of testing must wait for research on the techniques of evaluation to provide more effective ways of measuring the meaningful outcomes that are desired.

The Role of Research in the Improvement of Instruction

The purpose of instructional research is a functional one, namely, to provide the knowledge, both specific and general, that is needed to improve the instructional program. Research must supply the answers to take the place of the guessing which so frequently guides the program. Research must supply the evidence from which valid principles of teaching may be derived. These principles, in turn, must be the sources for deducing the proper applications to particular teaching situations. When research findings are interpreted and translated into applications, we have, not "action research," but research in action, producing actual functional outcomes.

The Contributions of Research Workers

In a brief talk at one of the sessions of the American Educational Research Association in February, 1950, W. A. Brownell presented some startling facts regarding research on arithmetic. His tabulations showed that, up to the end of the year 1945, there had been a total of 1,413 research reports on the subject of arithmetic, prepared by a total of 778 different authors. However, 615 of these authors had never reported more than one such study. Only 53 persons have reported more than three research studies in arithmetic. The lack of maturity in many of the published research reports on arithmetic is plainly related to the fact that so few persons have shown a consistent, long-time interest in contributing to the research literature.

The essence of research is a venture into the unknown. Yet it is difficult to induce students to risk a pilot study embodying techniques that are unproved. To do so is to invite criticisms which are the lot of all pioneer ventures. However, it is only by trying to discover new ways to probe into the thinking of pupils that one may hope to learn something about the meanings of arithmetic to the learner.

The Need for Research on the Learning Experiences of Children

Research in arithmetic is in danger of becoming crystallized into patterns which have proved useful in dealing with problems of learning through practice, in analyzing the content of textbooks and courses of study, in making surveys of social usage of arithmetic, and in comparing the effects of method *A* with method *B* as they apply to the teaching of various aspects of arithmetic. Some of these research studies were excellent, and much improvement in the teaching of arithmetic has resulted from them. But if teachers are to become sensitive to the meanings that arithmetic has for the children who are learning and if desirable habits of quantitative thinking are to be realized as outcomes, then techniques

of research must be devised which will reveal the thinking being done by children as they learn. There are few models to follow for this kind of research. The need is for ingenuity and inventiveness. We may need to learn to be patient with failures in some pilot studies, but we must not consent to be complacent about a lack of daring to undertake research to match the significant concepts regarding the teaching of arithmetic which have risen to prominence during the last two decades.

Planning for Needed Research Studies

It was the hope of the committee that a sampling of members of the profession who have already published studies on arithmetic might yield some proposals for research which would reflect the developing interest in meaningful outcomes. Accordingly, letters requesting suggestions regarding problems on which there is a recognized need for research were sent by members of the committee to a sample list including public school teachers, supervisors, and members of college and university faculties who had published articles or studies on the teaching of arithmetic. To facilitate comparative examination of the proposals, the respondents were asked to submit the statement of each problem on a blank form furnished by the committee which was designed to secure specific suggestions or information regarding certain factors considered essential to an appropriate plan for the research to be done. The factors listed included:

1. Significance of the problem in relation to present knowledge concerning arithmetic and to needs in the teaching of arithmetic.
2. Possible sources of data.
3. Suggested methods and techniques for carrying on the study.
4. Would the principal contribution be new knowledge about teaching and learning of arithmetic, new technique of research, or some other?

The twenty-one problems submitted are presented in the following section of this chapter. In each case the person contributing the proposal is identified. This will make it possible for others in the field who are interested in any one of these proposals to communicate with the contributor. The reader may judge the extent to which these proposed researches embody new ideas and techniques adapted to the concepts of a meaningful arithmetic and the extent to which they reflect a persistence of the interests and techniques which were common at the time the Twenty-ninth Yearbook was published. The committee confidently believes that the present yearbook will provide serviceable insights and guidance in aid of the endeavors of teachers and students who may desire to participate in the search for needed solutions to problems in the field of arithmetic such as those which are indicated by the proposals presented in this chapter.

Proposals for Research on Problems of Teaching and of Learning in Arithmetic

Proposal submitted by Harold Fawcett
Chairman, Department of Education
Ohio State University

Problem. A study of how children think about number, how concepts are developed, and how meanings are established.

Significance. Present knowledge does not provide us with information as to what goes on in the mind of a child when confronted with situations involving number concepts and number sense. The concept of number is a dynamic concept. How can the child's concept of number be extended most effectively to include fractions, decimals, and the like?

Sources of Data. Actual classroom situations under the direction of teachers interested in organizing their program in such a manner as to learn and record the thought processes of their students as their number concepts become enriched through guided experience.

Methods of Study. Teachers concerned with this study should be provided with competent help essential to the securing and recording of data accurately reflecting the thinking processes of the students. This would undoubtedly require much patience and many individual conferences with each student. If a child could be encouraged to talk as he works, to "think out loud," and if a recording could be made, the results would be helpful.

Contribution. The principal contribution would undoubtedly be "new knowledge" which could prove to be exceedingly helpful in improving the teaching of arithmetic.

Proposal submitted by Foster E. Grossnickle
Professor of Mathematics
New Jersey State Teachers College, Jersey City

Problem. The determination of the effectiveness of the use of manipulative materials in the teaching of arithmetic.

Significance. This problem is directly related to the question of teaching arithmetic meaningfully. Manipulative materials should enable the pupil to deal with things so that he can discover relationships among quantities more effectively than by use of symbolic and pictorial materials.

Sources of Data. The classroom should be used for an experiment to show the value of manipulative materials.

Methods of Study. Two equated groups at different grade levels, such as at Grades III and IV, or IV and V. One group should use a well illustrated textbook, preferably a book from a series published since 1940.

The other group should use the same text, but this group should supplement the text with the use of manipulative materials. The first group would not use these materials. A certain area of subject matter, such as carrying in addition, compound subtraction, understanding of fractions, and the like, should be selected for experimental purposes.

Contribution. If the results from the group using manipulative materials are significantly superior to those attained by the other group, these results may be regarded as proof that beginning a pupil at a low level of abstraction is an effective means of instruction in arithmetic.

Proposal submitted by W. A. Brownell
Dean, School of Education
University of California at Berkeley

Problem. An exploration of precomputational activities with common fractions. What are the possibilities not only for developing the various concepts of fractions themselves (e.g., ratio idea) but also for formulating a program with regard to related concepts (common denominator, reduction) and to an understanding of the operations?

Significance. There are plenty of data to show that children and adults alike have no skill with, or understanding of, common fractions. The reason may well lie in our failure to encourage meaningful approaches to computation.

Data and Method. The project is a long-term one, not to be satisfied by a study or two. Various programs need to be tried with children, and data secured by tests, interviews, etc., to identify sources of difficulty and of advantage. Our success in teaching the arithmetic of whole numbers rests upon a body of research studies like those here suggested for common fractions.

Contribution. Probably to knowledge about teaching and learning.

Proposal submitted by Harry G. Wheat
Professor of Education
West Virginia University

Problem. What are the attitudes toward arithmetic of pupils (*a*) of different abilities and (*b*) at the different grade levels who are succeeding in learning ways of self-instruction in the subject, that is, independent methods of answering the questions the subject raises?

Significance. Progress in learning arithmetic relates to attitude. It is important to discover just how far favorable attitude is a result of learning as well as a determining stimulant.

Sources of Data. The classrooms of teachers who make well-planned effort to teach pupils methods of self-instruction.

Method of Study. (*a*) The teacher (experimenter) leads to a new topic or process through a review and new view of what pupils have already learned, helping them to fit different learnings into a similar pattern—the pattern of the number system, for example—and making clear the new kind of question the new topic raises. (*b*) The teacher encourages suggestions by pupils about the possible ways of answering the new question, leading them to the single method of proven merit which the teacher wishes the pupils to learn and use. The teacher is careful that each pupil is clear about the steps to take in the method of work. (*c*) The teacher provides the questions, and each pupil seeks his own answers; the teacher checks the speed of learning of each pupil. (*d*) The teacher, through observations, questions and answers, discussion, and otherwise, notes the attitude of each pupil. (*e*) The teacher makes a rough classification of attitudes according to qualities and amounts and correlates with the pupil's ability and his present grade level.

Contribution. The chief contribution would probably be (*a*) new knowledge about the teaching and learning of arithmetic and (*b*) some slight basis of a change in attitude of teachers toward the subject in the school.

Proposal submitted by C. Louis Thiele
Divisional Director, Exact Sciences
Detroit Public Schools

Problem. To compare the relative effects of early and of late introductions to the use of algorisms as such. The problem could be related to the learning of any one of the larger skills of arithmetic, as, for example, of division by two-place divisors, of the addition, subtraction, multiplication, or division of common or decimal fractions, of simple ratio, or of percentage.

Significance. Such experimentation should contribute to present knowledge regarding the part that meaning plays in the mastery of arithmetic.

Sources of Data. Paired groups might be used in an experimental study.

Method of Study. One group would be inducted into the use of algorisms almost immediately after new processes had been performed concretely and symbolized. The other group would continue to perform on the thought level without the benefit of algorisms until the number situations became too complex for solutions on the thought level. For example, in the study of percentage, many simple problems could be solved involving the three cases of percentage with only the knowledge that per cent means "by the hundred" or "out of every hundred" before algorisms are introduced.

Contribution. New knowledge about the teaching and learning of arith-

metic. Light might be thrown on problems related to time distribution, allotment of arithmetic topics, growth in arithmetical learning, and individual differences in learning arithmetic.

Proposal submitted by Leo J. Brueckner
Professor of Education
University of Minnesota

Problem. The development of resource units for teachers to enable them to vitalize instruction in arithmetic and to supply them with up-to-date materials dealing with social applications of arithmetic.

Significance. An analysis of newer courses of study reveals that marked emphasis is being placed on the social phase of arithmetic. However, there is a dearth of well-organized information for teachers about many of the topics that are suggested. There is a demand from many teachers for resource units which will make available the needed information.

Sources of Data. The units to be made available should be developed under competent guidance in the classrooms and evaluated to show their effectiveness. Teachers have the right to ask for evidence that the units proposed or reported can be adapted to a particular level of schoolwork and also for evidence of the results secured when the unit was actually tried in the classroom. Many more units should be prepared than any classroom teacher can use, thus requiring selection and adaptation to local conditions.

Methods of Study. The development of resource units can be done under the direction of curriculum bureaus, or by groups of teachers interested in the improvement of instruction in arithmetic. The facilities of a well-supplied library should be available. The appraisal of the outcomes of such units will require also the development of a variety of types of appraisal instruments that are now lacking.

Contribution. The principal contribution of this project would be the development of a series of rich instructional units that would do much to improve the curriculum and methods of instruction in the field of arithmetic. Also, many workers would have valuable experience in modern curriculum-making procedures, since inevitably teachers in various localities would be stimulated to develop local units growing out of emerging problems and situations that would be realistic and vital.

Proposal submitted by Ben A. Sueltz
Professor of Mathematics
State Teachers College, Cortland, New York

Problem. To establish methods of evaluating a pupil's ability to use arithmetic in functional situations. A functional situation is one that

is encountered in experience and may present itself in writing, in speech, in a visual or kinesthetic impression, or in a combination of these.

Significance. The problem is important because arithmetic aims to be meaningful to pupils and to be functionally useful to them. Up to date, our research in evaluation has dealt largely with printed media and with casual observation. The need is for a type of evaluation that deals with real materials in real situations.

Data. Sources of data depend upon setting up new methods of testing and evaluation which must be used experimentally.

Method of Study. To carry on this study, the researcher must develop his own tools of measurement, establish their validity, and use them in a comprehensive study. Several people have experimented in this field but little of significance has been done.

Contribution. This study would yield new procedures and techniques of measurement and evaluation.

Proposal submitted by John R. Clark
Professor of Education
Teachers College, Columbia University

Problem. A comparison of "*deductive*" and "*memory*" *learning of basic addition and subtraction facts.* For example, should pupils *learn* that 8 and 5 are 13 by thinking "8 and 2 are 10, and 3 more are 13" or by the conventional type of repetitive drill?

Significance. If research outcomes favor the first method, great labor cost for pupil and teacher would ensue.

Sources of Data. Paired groups—an experimental study.

Methods of Study. Use "oral analysis-deduction" with one group and conventional memory drills with the other. Test transfer effects to higher decade facts.

Contribution. New knowledge.

Proposal submitted by Harold E. Moser
Maryland State Teachers College, Towson

Problem. We know from studies conducted by Brownell that third-grade children profit more from subtraction taught meaningfully by the decomposition method than from equal additions taught meaningfully. Johnson and others have shown that the process of equal additions is more efficient at the adult level. Where, in the arithmetic curriculum, can we teach children to change from decomposition to equal additions with a minimum of confusion and interference?

Significance. The teaching of arithmetic has long been dominated by the false doctrine of "teach in the form in which learning is to be used in

life." This practice is contrary to the concept of learning as reorganization of experience. We never have fully explored the range of reorganizations a child can make helpfully as he carries his thinking to higher mental levels.

Sources of Data. Study should be made in schools where the children have been taught the meanings of the number processes and the rationale of the algorisms. The grades surveyed might range from Grade V to Grade VII or VIII.

Methods of Study. It would be necessary to use a control group and an experimental group at each grade level studied. The control group would be given practice with decomposition equal in amount to the training given the experimental group with equal additions. We may assume that the control group would make the greater gain at any grade level, but as children in the experimental group experience less and less interference their scores will show closer and closer approximations to gains made by the control group.

Contribution. Valuable information about the nature of the learning process.

Proposal submitted by Edwina Deans
Assistant Professor of Education
Teachers College, University of Cincinnati

Problem. The value of the interview technique as a testing procedure to help determine the thought processes of pupils has been rather definitely established. Is it not possible that the interview technique also has diagnostic value as an instructional procedure? It is conceivable that research might determine the values of continuous day-by-day group and individual interview techniques for revealing processes of thinking in arithmetic.

Significance. Good teaching necessitates (*a*) that the teacher will know the learning level of children, where they are and what they are ready for; (*b*) that children themselves know where they are and what the next steps in learning are for them; (*c*) that children learn continually to evaluate their learning procedures, to slough off less mature ways of obtaining answers and take on more mature ways.

Sources of Data. An intensive study of one group of children would offer much valuable information on this problem. A grade level at which children are concerned with learning the fundamental processes would be appropriate. It would be desirable for the study to extend throughout the year, or for a longer period if the teacher follows the same group of children for more than one year. Diary records of instructional procedures as well as accurate reports of children's responses would need to be kept.

Methods of Study. (*a*) Interviews with individual children at the beginning of the investigation to determine methods of thinking or levels of performance for each child. (*b*) An instructional program designed to stress ways of thinking. Different children will often be asked to describe their methods of obtaining answers as group instruction progresses. As methods are described, they are evaluated by the children under the direction and guidance of the teacher on the basis of certain criteria, such as (1) promise for getting the correct answer, (2) economy in time and effort, and the like. (c) Interviews with individual children at the end of the investigation to determine growth in maturity of thought processes. Some measure of ability to transfer learning to untaught phases of the processes would be desirable.

Contribution. The improvement of classroom instruction, and the enrichment of knowledge about the teaching and learning of arithmetic.

Proposal submitted by Vincent J. Glennon
Assistant Professor of Education
School of Education, Syracuse University

Problem. The optimum ratio of class time to be spent on the development of understandings (meanings) and the development of computational skill or facility.

Significance. The above problem is one of the most significant problems in the teaching of arithmetic. We generally agree that we should teach meanings but do not know how much time can be taken from the drill periods to teach meanings. It may be found that children can learn the skills better by spending less time on drill and more time on meanings. In addition, they will acquire the meanings that will make the skills permanent and useful.

Sources of Data. Using grades one through six, the groups should be equated according to mathematical understandings, computational and problem-solving skills, mental ability, reading ability, and attitude toward arithmetic.

Methods of Study. Select several understandings that are basic to commonly used computational processes. A teaching period of convenient length, say six weeks, with the same teacher for all groups. Group A spends five weeks on meanings, one week on drill. Group B spends four weeks on meanings, two weeks on drill. Group C spends three weeks on meanings, three weeks on drill. Group D spends two weeks on meanings, four weeks on drill. Group E spends one week on meanings and five weeks on drill. (Two more groups could also be used—one with no weeks on meanings and six weeks of drill; the other with six weeks on meanings

and no weeks of drill). At the end, the groups would be compared on the basis of understanding and ability to compute.

Contribution. Knowledge about the teaching and learning of arithmetic.

Proposal submitted by Robert H. Koenker
Associate Professor of Education and Director of Graduate Studies
Ball State Teachers College, Muncie, Indiana

Problem. To discover the value, if any, of a rich concrete-number readiness program in the kindergarten and Grade I for later success in arithmetic.

Significance. Reading-readiness programs have proved of great value, and it seems logical that a companion arithmetic-readiness program would be equally as valuable. At present most schools do very little work in arithmetic in the kindergarten and Grade I. Are they justified in placing so little stress on arithmetic?

Sources of Data. Paired groups of school children.

Methods of Study. At the beginning of the kindergarten year, select a control and an experimental group. A number of schools should be used to give a large number of cases. During kindergarten year and Grade I the control group would receive the regular program, while the experimental group would receive the regular program plus a rich arithmetic-readiness program. These groups should be followed through the elementary school so as to evaluate the effects of the experimental factor on arithmetic achievement.

Contribution. Since the value of number work in the kindergarten and Grade I is a controversial matter, the findings of this study would have a bearing on the arithmetic curriculum and future research.

Proposal submitted by R. L. Morton
Professor of Education
Ohio University, Athens, Ohio

Problem. The extent to which learning is accelerated or impeded by the language of the teacher or the instructional materials. For example, is the pupil who understands what is meant by, "Divide the candy bar into thirds," temporarily blocked in his learning by, "Divide 3 into 15," when confronted with the situation, $3\overline{)15}$? If he were as intelligent as his teacher should be, might he not reply, "I can divide 15 into 3's but it is not possible to divide 3 into 15?"

Significance. Teachers have a tendency to invest familiar words with new and specialized meanings without thought of the temporary confusion in learners. A "reasonable answer" may at first suggest to the pupil

something that is cheap, especially if the answer happens to represent money. So far as I know, this aspect of language difficulty has not been the subject of systematic study.

Sources of Data. Analysis of instructional materials, observations of teachers, interviews with pupils.

Contribution. New knowledge about the teaching and learning of arithmetic.

Proposal submitted by Herbert T. Olander,
Professor of Education
University of Pittsburgh

Problem. To what extent can primary-school children discriminate between reversals among the simple addition number combinations.

Significance. If the majority of primary-school children, particularly in Grades I and II, should be found not to possess the necessary space discrimination (or at least did not make the reactions) to recognize that one reversal is different from the other, is there much purpose in teachers presuming, at least as far as the child is concerned, that one reversal is different from the other? For example, do second-grade children notice that in 2 + 3 the "2" precedes the "3," or that in $\frac{2}{3}$ the "2" is above the "3"?

It is a well-known fact that young children have a relatively poor orientation to space or space relationships. Note, for example, how they often draw a man horizontally instead of in an upright position. This has led me to believe that a small child may see no difference between reversals such as 4 + 3 and 3 + 4, whether these are presented horizontally or vertically. That is, the children, or at least many of them, may react only to the elements present and not to their orientation in space. If that is true, I can see a possible explanation for the tremendous transfer I found in addition combinations in my thesis problem at the University of Chicago. However, I am at a loss to explain a similar transfer I found in subtraction, in which case this reasoning does not seem to apply. The transfer I found in subtraction leads me think that the children perhaps did generalize, after all.

Sources of Data. Reactions of primary-school children.

Methods of Study. Experimental study. Illustrative techniques: Show the children addition number combinations on flash cards and then have them (*a*) copy these combinations on paper, or (*b*) after flashing a particular combination like 2 + 3 on a flashcard, have the children check this combination among a list of half dozen or so combinations, among which appear both 2 + 3 and 3 +2.

Contribution. Contribution to knowledge about children's learning of arithmetic.

Proposal submitted by George H. McMeen
Assistant Professor
New Jersey State Teachers College, Newark

Problem. How far should the teacher of arithmetic carry his process of rationalization in teaching the fundamental operations of arithmetic? For example, in subtraction of large numbers, is it advisable to rationalize the decomposition process in toto?

Significance. The teacher and textbook writer could save much time and effort if they knew at what point the pupil can transfer his learning to meet the new situation or whether it is relatively unimportant for him to make such rationalizations. This problem is closely related to the problem of how rationalization by the pupil facilitates related learning.

Sources of Data. Practice exercises involving different degrees of decomposition. Writers in the field differ markedly with regard to this problem. For example, some writers would rationalize the "borrowing" process in subtraction only out to the hundreds place, others to thousands, and others still farther.

Methods of Study. The control-group versus the experimental-group approach extended over a wide interval of time to test the value of rationalization in later work, perhaps even to the adult level.

This problem is too big for one person alone to attempt and should be studied by a large group with ample funds.

Contribution. At the present time this problem is only in the "opinion" stage, but I would say that it would contribute new knowledge or provide confirmation of assumptions about the teaching and learning of arithmetic.

Proposal submitted by Robert L. Burch
Assistant Professor of Education
Boston University

Problem. To determine at several points in the elementary-school arithmetic program the efficacy of utilizing the ideas of place value instead of mechanical rules and generalization.

Significance. Through utilizing the concept of place value, it is possible to rationalize such procedures as aligning addends, placing partial products, correctly placing quotient figures, locating decimal points, and so on. If it can be shown experimentally that the use of this underlying idea is an adequate means for carrying out a number of procedures that would otherwise require the learning of many rules, the research will significantly reduce the memory load in the arithmetic program and will add further evidence to substantiate the value of the meaning approach to arithmetic.

Sources of Data. (*a*) A series of parallel-group experiments at different grade levels which involve teaching the "place-value approach" to experimental groups and the "rules" to control groups. (*b*) Remedial programs for groups that are having trouble with the fundamental processes with whole numbers and decimals.

Methods of Study. (*a*) Set up parallel-group experiments at points where pupils are being taught new topics that lend themselves to either the place-value approach or the specific-rule approach (locating partial products, placing quotient figures, and so on). Be certain that the pre-experimental testing includes readiness tests for the new topic as well as other means for determining initial status and for balancing groups. Regardless of whether the teaching approach is through the ideas of place value or through the teaching of rules, the pupils should be guided to discover the new materials in so far as possible. Postexperimental evaluation should not just include measures of rate and accuracy. Rather, the evaluation program should include reports of pupils' oral performances, teachers' opinions of pupil interest in the topic, and measures of long-term retention and ability to transfer the new understandings and skills.

(*b*) If large enough remedial groups who had learned the topics through rules could be located, it would be worth while to compare the efficiency of more teaching which involves rules as contrasted to the use of the new approach of the place-value concept. If parallel groups were not available, it would still be worth while to find the effectiveness of teaching the place-value approach to pupils who had been unable to manage such topics as locating partial products, placing quotient figures, or determining the placement of decimal points through use of memorized rules.

Contribution. In the series of research studies suggested above, the principal contribution would be in the area of the teaching and learning of arithmetic.

Proposal submitted by F. Lynwood Wren
Professor of Mathematics
George Peabody College for Teachers, Nashville

Problem. A study of the contrast between (*a*) an estimation technique with emphasis on place value and (*b*) current methods for obtaining quotient figures and placing them properly in division. The estimation technique would call for rounding both divisor and dividend to the same denomination at each step in the division. For two-digit divisors, they would be rounded to the nearest ten.

Significance. It might point the way toward making the teaching of division more meaningful and less mechanical than it now is.

Sources of Data. (*a*) History of mathematics to show the significance of place value in the evolution of our number system; how it has aided in giving simplicity and universality to our concepts and uses of number. (*b*) Review of educational literature to point up the strong points and weak points of division methods now in use; in particular, to point out the major sources of student difficulty. (*c*) Careful experimentation to measure the contrast.

Methods of Study. This study would have to be a long-term experiment with carefully matched groups. Their progress would have to be followed at least from the third grade through the sixth grade.

Contribution. The principal contribution would be new knowledge about the teaching and learning of arithmetic.

Proposal submitted by Sina M. Mott
Associate Professor of Preschool Education
Southern Illinois University, Carbondale

Problem. What arithmetic concepts are being used by five-and six-year-old children?

Significance. It is basic to the whole problem of grade placement, to the problem of workbooks and their contents, and to the problem of methods.

Sources of Data. (*a*) Findings in the literature of research. (*b*) Analysis of the arithmetic concepts being used by the five- and six-year-olds in the home and in the school.

Methods of Study. Preliminary Study—(*a*) Analyze the material already gathered in the research literature. (*b*) Secure the co-operation of ten teachers and thirty parents. Have them record all the concepts used by the five- and six-year-olds, the teachers for ten days (two school weeks) and the parents for ten days beginning on a Friday and ending the second Monday. (*c*) Analyze and tabulate this material. (*d*) Build a check list from these data. *General Study*—Use the check list developed by the above technique over a wider range—economic, social, and geographic—of homes and schools. Tabulate results, and build syllabus.

Contribution. The study will yield a body of knowledge containing the arithmetic concepts used by five- and six-year-olds. These data might well be used in building the curriculum for kindergarten and first grade. It might also be the basis for experimentation as to the best methods of teaching or presenting arithmetic concepts.

Proposal submitted by W. J. Osburn
Professor of Education
University of Washington, Seattle

Problem. Validation of techniques for bringing out arithmetic meanings. Validation of techniques on long division.

Significance. Useful techniques seem significant in connection with the prevailing tendency to emphasize meaning in arithmetic.

Source of Data. Pupil performance.

Methods of Study. Ask teachers to use the techniques with the pupils.

Contribution. New knowledge about teaching and learning arithmetic.

Proposal submitted by John W. Dickey
Associate Professor of Mathematics and Education
New Jersey State Teachers College, Newark

Problem. Methods of improving the mental hygiene of mathematics.

Significance. Much of the failure in mathematics and the failure to choose courses in mathematics at high-school and college levels are probably due to bad emotionalized attitudes.

Sources of Data. Reports of students and teachers.

Methods of Study. Interview methods.

Contribution. Probably new knowledge related to the teaching and learning of mathematics.

Proposal submitted by Elsie M. Weber
Denver, Pennsylvania

Problem. The construction of appropriate tests to measure resourcefulness in number situations.

Significance. Most teachers subscribe to the point of view that learning of arithmetic should be meaningful and not mechanistic. Meaningful learning emphasizes discovery and problem-solving instead of rote learning based on repetition.

Data and Method of Study. The measures of growth used in the past, such as the ability to respond quickly and accurately to number facts, are inadequate to measure the elements in a meaningful program. A measure of speed and accuracy does not test all of the abilities which teachers seek to develop in a meaningful program. This is especially true of the aim which attempts to make children resourceful in the discovery and use of number relationships. For want of a better term, the development of this ability may be placed in the category of "number sense" or "meaningful arithmetic." New measures of such development need to be constructed and tested experimentally.

Contribution. New techniques relating to the ability to attack and to solve new problems successfully. How is it possible to measure resourceful behavior?

INDEX

Abstract materials, use of, in the teaching of generalizations, 151–53

Accrediting association, certification standards recommended by, 221–25

Acquisition of basic meanings in arithmetic, 79

Addition with decimal fractions, 93

Age of pupil in relation to number work, 59–64

Analysis of instructional procedures in arithmetic, 128–35

Analysis of wholes in developing meaning of fractions, 88–89

Answers to number questions, determination of, 27–29

Application of theories of learning to the teaching of arithmetic, 255–56

Approximate numbers, use of, in measurement, 245–46

Arithmetic: applying theories of learning to, 255–56; learning activities to improve instruction in, 256–57; organizing experimental teaching in, 257–61; place of, in school curriculum, 6; preparation of teachers of, 225–27; in relation to social institutions, 272–78; as response to human needs, 269–72; teacher's understanding of content of, 253–55; a theory of, for schools of today, 278–81; as a way to think, 22

Arithmetic experiences: emphasis on meaningfulness of, 77; as method of teaching, 78

Arithmetic tests: evaluation of, in professional writings, 194–96; list of, 200–201; recommendations for improvement of, 196–200

Arithmetical abilities required in other school subjects, 9–10, 14

Arithmetical concepts: ability of pupils to deal with, 12–14; teacher's understanding of, 235; use of, in other fields of study, 10–12

Arithmetical meanings: emphasis on, in current learning theories, 149–51; relation of, to motivation, 151; types of, 149–50

Arithmetical operations, drill on fundamentals of, 239–41

Association theories of learning, influence of, on organization and teaching of arithmetic, 145–46

Attitude, as a factor in learning arithmetic, 23

Background course in mathematics for elementary teachers: need for, 232–35; suggested organization of, 234–35

Basic meanings, acquisition of, in middle grades, 79

Basic operations in arithmetic, 239–41

Brownell, W. A., 6–7, 149–50, 283

Cardinal number, pupil's understanding of, 61–62

Certification of elementary teachers, state requirements for, 203–5

Child growth and development in arithmetic, 54–58

Classroom, equipment of, for arithmetic instruction, 71–74

Classroom experiences, use of, as problem-solving situations, 114–15

Commission on Postwar Plans, Second Report of, 19–20, 220, 277–78

Common-denominator method of dividing with fractions, 90–92

Common-fraction meanings, study of, in middle grades, 86–88

Comparison of numbers, methods used in, 110

Competence in arithmetic thinking, as safeguard against frustrations, 57–58

Computational skill, values of, in high-school program, 112

Content of arithmetic: in primary grades, 58–74; organization of, 150–51

Contrasts in arithmetic and other content subjects, 1

Contribution of arithmetic to other fields, 14–19

Contributions of other fields to arithmetic, 15–20

Contributions of yearbook to the teaching of arithmetic, 2–4

Curriculum patterns, as related to arithmetic, 8

Curriculum planning committee for arithmetic in the elementary schools of Philadelphia, 262

Decimal-fraction multipliers, 93–95
Decimal fractions: addition with, 93; meaning of, 92–93; subtraction with, 93; use of, in division, 95–96
Descriptive function of arithmetic, 107
Differentiated curriculums in teacher education, 206–8
Division with more than one digit, 84–86
Drill, role of, in field theory of learning, 147–49
Drill exercises, use of, at different grade levels, 131
Drill method, characteristics of, 78

Elementary-school teachers: background course in mathematics for, 232–35; state requirements for certification of, 203–6
Equipment for teaching arithmetic in primary grades, 71–74
Evaluation of formulas, importance of, in education of teachers of arithmetic, 247
Experimental teaching in arithmetic, 257–61
Exploratory procedures in connection with new problem situations, 132–35

Fawcett, Harold P., 105
Field theories of learning: influence of, on organization and teaching of arithmetic, 146–53; role of drill in, 147–49
Formulas, ability of arithmetic teacher to evaluate, 247
Fraction divisor, methods of computing with, 90–92
Fraction multiplier, use of, in multiplication, 89–90
Fractions: evolution of meaning of, 271; problems involved in the teaching of, 241–44
Frustrations, competence in arithmetical thinking as safeguard against, 57–58
Fundamental number processes: meanings of, 64–66; teaching meanings of, in primary grades, 66–68; use of, in problem-solving, 66
Fundamental operations, separate drills on, 239–41
Fundamental skills, use of, in high-school mathematics, 111–14

Generalizations, use of abstract and concrete materials in the teaching of, 151–53
Grade placement in relation to content of arithmetic, 59–64

Hartung, Maurice L., 77
High-school arithmetic: basic concepts pertaining to, 106–15; content and aims of, 104–6
Hilgard, Ernest R., 145
Historical development of arithmetic, 235–36, 269–72

Illustrative programs for improving instruction in arithmetic, 262–68
Importance of arithmetic in school curriculum, 3
Improvement of instruction: examples of programs for, 262–68; relation of supervision to, 251–62; role of research in, 283
Incidental learning, value of, for arithmetic, 18–19
Independent work, importance of, in pupil's progress in arithmetic, 30
Individual differences in pupil achievement, 129–30
Instruction in arithmetic: conflicting theories of, 78; initial approaches to, 120–28; function of supervision in relation to, 251–62; role of research in the improvement of, 283
Instructional materials: classification of, 161–63; illustrations of the use of, 164–72; nature and uses of, 155–56; in relation to instructional program, 163–64
Instructional procedures in arithmetic, analysis of, 128–35
Integers, properties of, 238–39
Integrated curriculums, relation of arithmetic to, 17–18

Junior high school arithmetic: basic concepts pertaining to, 106–15; content and aims of, 104–6

Knight, F. B., 143

Laboratory method, use of, in teaching arithmetic, 157–58
Language of arithmetic, 23–24
Language of mathematics, understandings essential to use of, 108–14
Learning in arithmetic: assumptions with regard to, 78–81; influence of pattern of thinking on, 24; relation of instructional materials to, 155–64
Learning experiences of children, need for research on, 283–84

Learning theory, relation of, to instruction in arithmetic, 140-42
Liberal-arts colleges, preparation of arithmetic teachers in, 225-27

McConnell, T. R., 144
Manipulative materials, use of, in teaching arithmetic, 173-76
Mathematical phase of arithmetic, 107-8
Mathematics, background course in, for elementary-school teachers, 232-35
Maturation, relation of, to learning, 153
Meaning, as factor in pupil's progress in arithmetic, 30
Meaning of decimal fractions, 92-93
Meaning as a factor in motivation, 151
Meanings, purposeful guidance of acquisition of, 80
Meanings of numbers, 60-64
Measurement: the arithmetic of, 244-46; definition of, 109-10; different systems of, 245; understanding process of, 245; use of approximate numbers in, 245-46
Methods: new ideas about, of learning and teaching arithmetic, 2; of teaching arithmetic in middle grades, 78-81; of teaching in primary grades, 58-74
Middle-grade arithmetic: content of, 81-101; critical nature of, 76
Motivation, relation of, to emphasis on meaning, 151
Motive, as a factor in learning arithmetic, 23

National Commission of Teacher Education and Professional Standards, 204
National Council of Teachers of Mathematics, 116, 143, 220
New emphases in the teaching of arithmetic, 1-2
New situations in problem-solving, 37-38
Notation, importance of understanding of, 62-63
Number charts, use of, in teaching number facts, 147-48
Number experiences, utilization of, 69-71
Number names and number signs as the language of arithmetic, 23-24
Number symbols: generalization of, 79-80, reading and writing of, 62; relation of, to objects, 79
Number system: developing an understanding of, 154; learning the use of, 24; positional notation in, 236-38
Number-thinking: as a mental process, 25-26; practical situations, as a factor in, 31-33; single road of, 31-32; steps in, 31-32

Operations, concept of, in arithmetic, 110-11
Ordinal number, importance of, 61

Part-whole relationships, three-fold interpretation of, 100-101
Pattern of thinking required in arithmetic, 39-43
Percentage, meaning of, 96-98
Pictorial materials, use of, in teaching arithmetic, 176-77
Place value concepts, gaining an understanding of, 62-63
Positional notation in number system, 236-38
Practice: in problem-solving, 37; relation of, to emphasis on meaning, 153
Preparation of teachers of arithmetic, 3-4
Probability, importance of, in education of teachers of arithmetic, 247
Problem exercises, locations of, 34
Problem-solving: importance of, in junior and senior high school program, 113-14; improving pupils' competence in, 139; new situations in, 37-38; practice in, 37; processes involved in, 33-39; use of fundamental processes in, 66
Programs for improving instruction in arithmetic: in city school system, 262-64; in a small elementary school, 266-68; in urban-rural school system, 264-66
Projection materials: use of, in arithmetic instruction, 167, 177-84
Proposals for research studies on problems of teaching and of learning in arithmetic, 285-97
Psychology of arithmetic, changing conceptions in, 143-45
Pupil action as basis of learning in arithmetic, 26
Pupil's thinking, quality of, as outcome of teaching, 29-30

Ratio, meaning of, 98-100
Readiness: attainment of, for arithmetic, 54-55; factors involved in, for learning arithmetic, 156-57; for number experiences, 55-57
Reciprocal method of dividing with fractions, 90-92
Recognition of number questions in learning arithmetic, 27

Reference units, use of, in interpreting quantitative statements, 14
Research on arithmetic, relation of, to the improvement of instruction, 282–84
Research studies in the field of arithmetic: need for, in relation to new concepts of teaching, 283–84; selected list of proposals for, 285–97
Rote counting, early experience in, 60–61

Sample lessons in different methods of teaching arithmetic, 122–28
Schorling, Raleigh, 18–19
Science as source of problem-solving situations, 115
Scientific Research Board, 220
Scope of arithmetic in elementary school, 6–7
Senior high school mathematics, need for more diversified program in, 115–19
Sequences of learning activities in arithmetic, 39–50
Single road of number-thinking, 31–32
Social basis of arithmetic, 3
Social-economic sources of problem-solving experiences, 115
Social institutions, effects of, on arithmetic, 273–78
Social phase of arithmetic, 107–8
Standardized tests, use of, in arithmetic, 190–94
Standards for certification of teachers, 221–25
Steps involved in learning arithmetic, 43–51
Steps in learning number, 156–61
Steps in number-thinking, 31–32
Subtraction with decimal fractions, 93
Supervision, new concept of nature and role of, 251–53
Supervisor, role of, in the improvement of instruction, 251–62
Symbolic representation of experience in learning number, 159
Systematic teaching of arithmetic, reasons for, 17–21

Teacher education, differentiated curriculums in, 206–8
Teacher guidance of pupil activities in arithmetic, 26–27
Teacher preparation in arithmetic, 3–4
Teachers of arithmetic, preparation of, 225–27
Teachers' colleges: admission requirements of, 208–9; background in mathematics for entering, 209–13; graduation requirements of, 216–20; training teachers of arithmetic in, 209–29
Teaching arithmetic, present concepts of, 186–87
Testing instruments and practices in arithmetic, 187–94
Tests, use of, in instruction, 137–38
Textbook tests, use of, in arithmetic, 189–90
Textbooks, use of, in different situations, 135–37
Theories of learning, classification of, 144–45
Thorndike, E. L., 145

Understanding number relations, methods of developing, 148–49
Units of measure, historical development of, 270–71

Van Engen, H., 87
Verbal representation of quantitative situation as a step in learning number, 159
Visual aids, use of: in instruction, 138–39; in teaching arithmetic, 130–31, 172–84

Wheeler, R. H., 143, 144
Whole-number meanings, importance of, in middle grades, 82–83

Yearbook: aims of, 2–4; organization of, 4–5

Zero: influence of, on operational thinking with numbers, 271–72; as placeholder, 63–64
Zero concept, limited use of, in primary grades, 63–64

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